Sharif Quantum Information Group

# Topology and Quantum Computation

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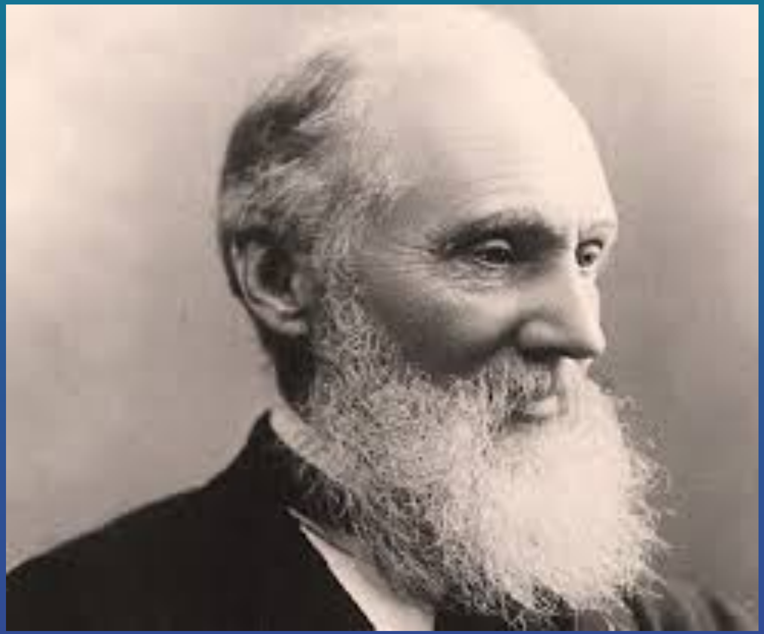
Sharif University of Technology, Iran.

Sharif Colloquium
1395

# The origins of Knot Theory

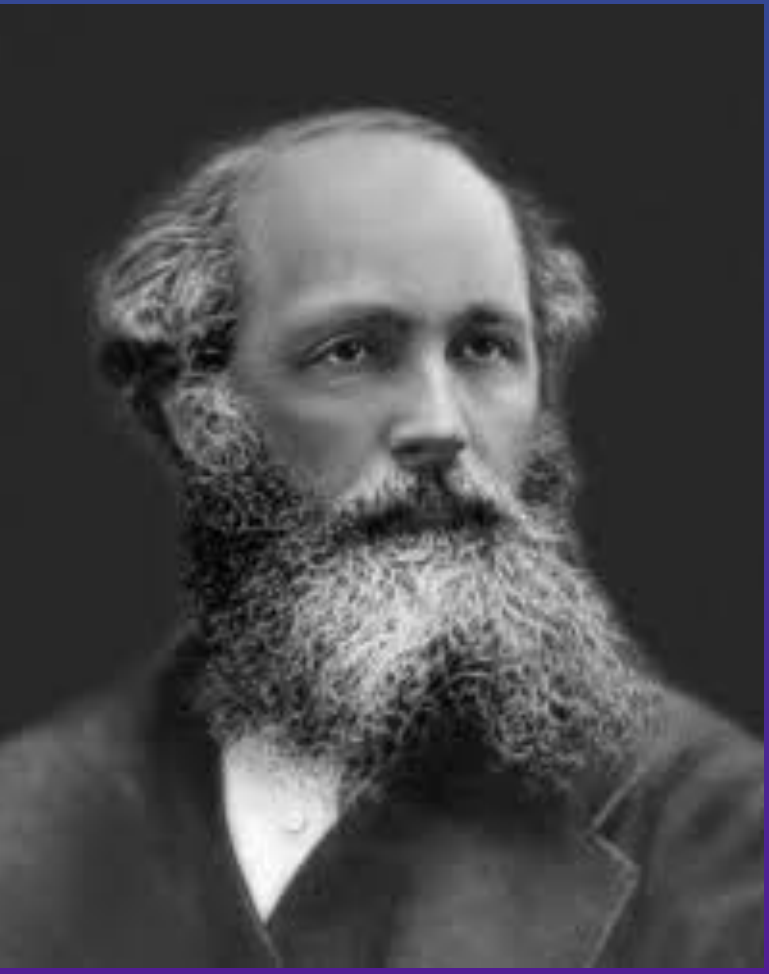


Kelvin

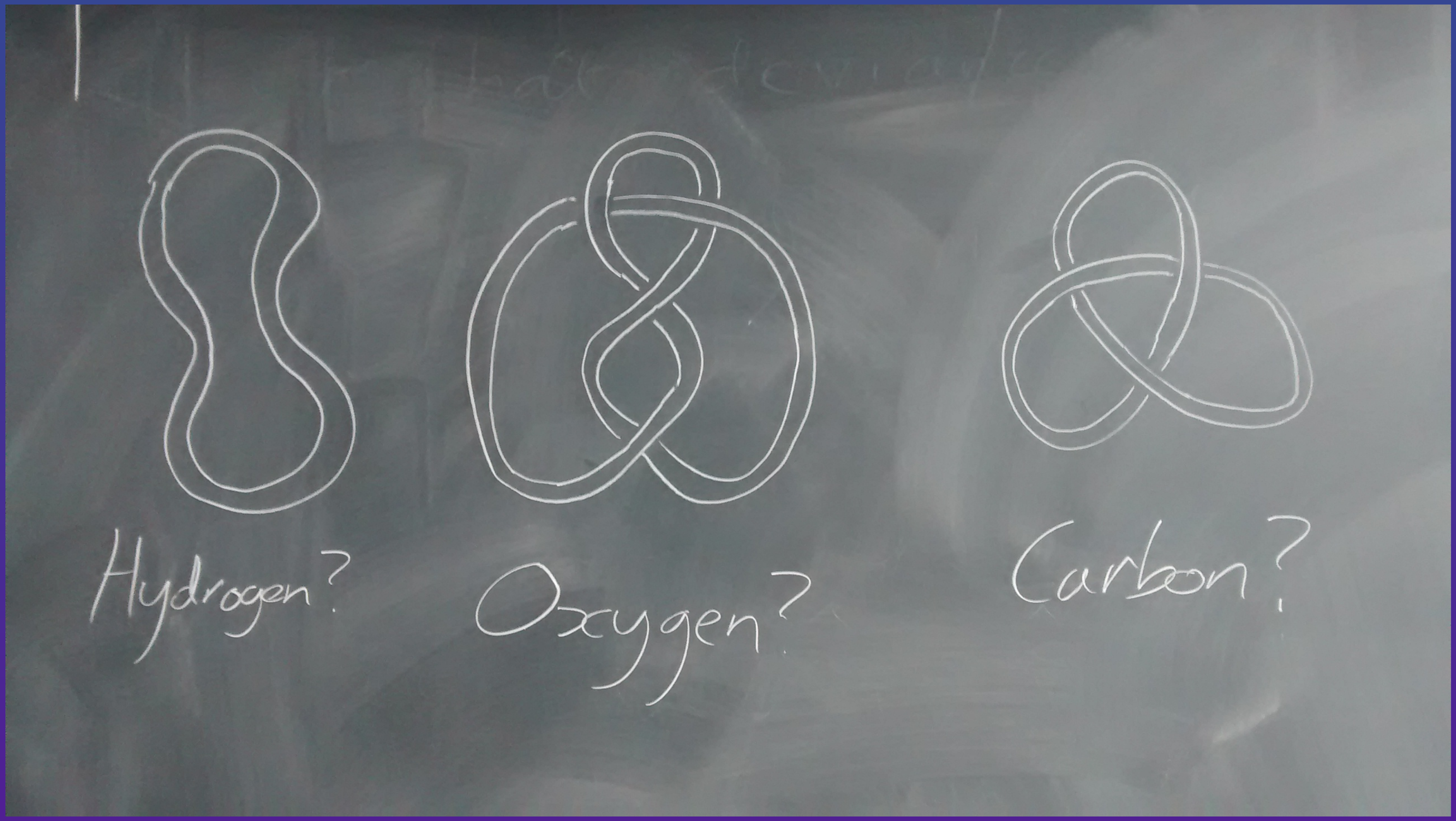
Vortex lines in fluids are stable!

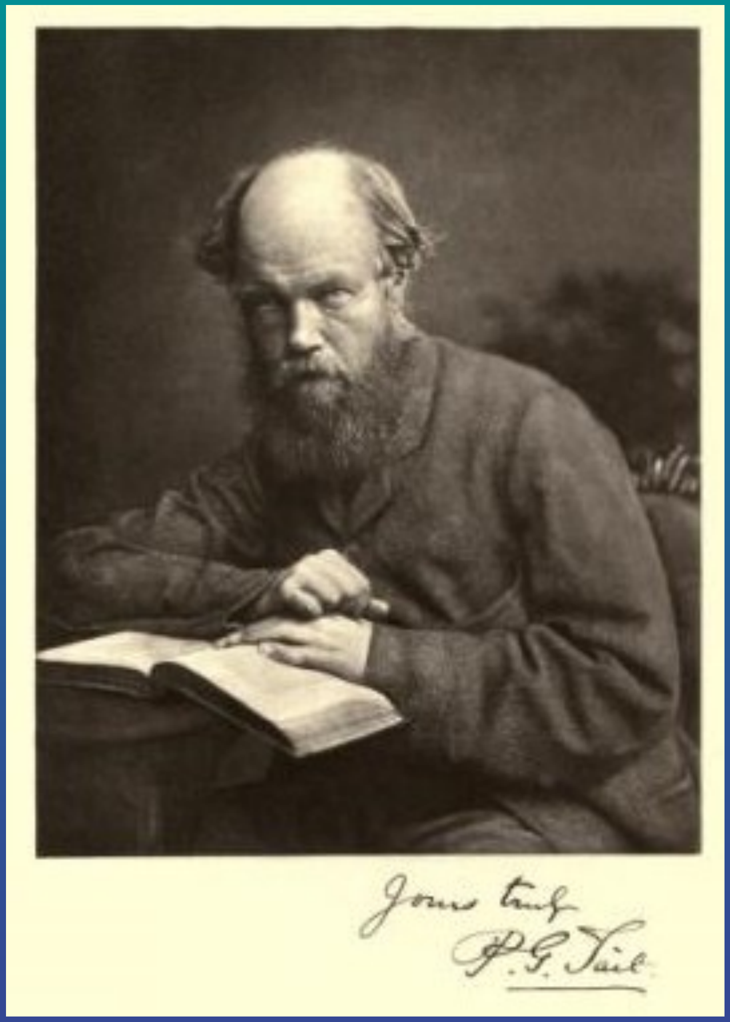
Atoms are Knots in Ether!

They are stable and take many forms.



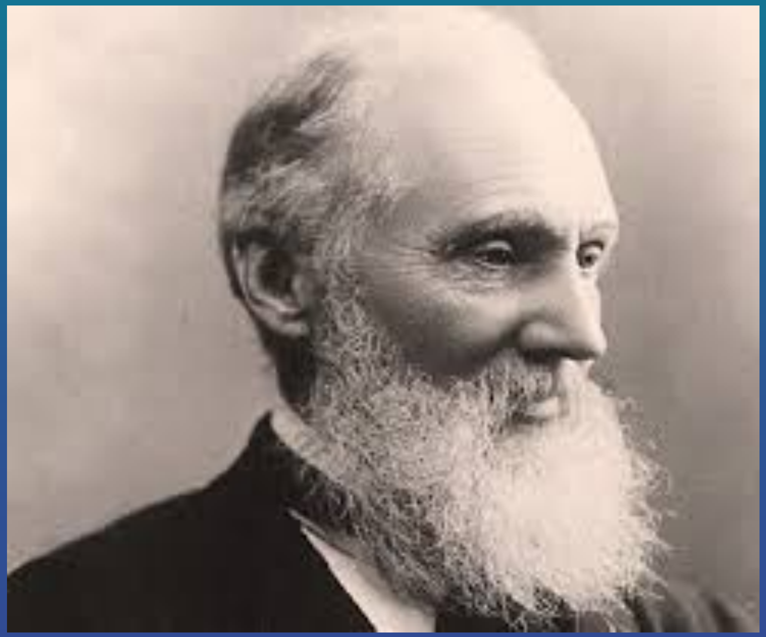
Maxwell





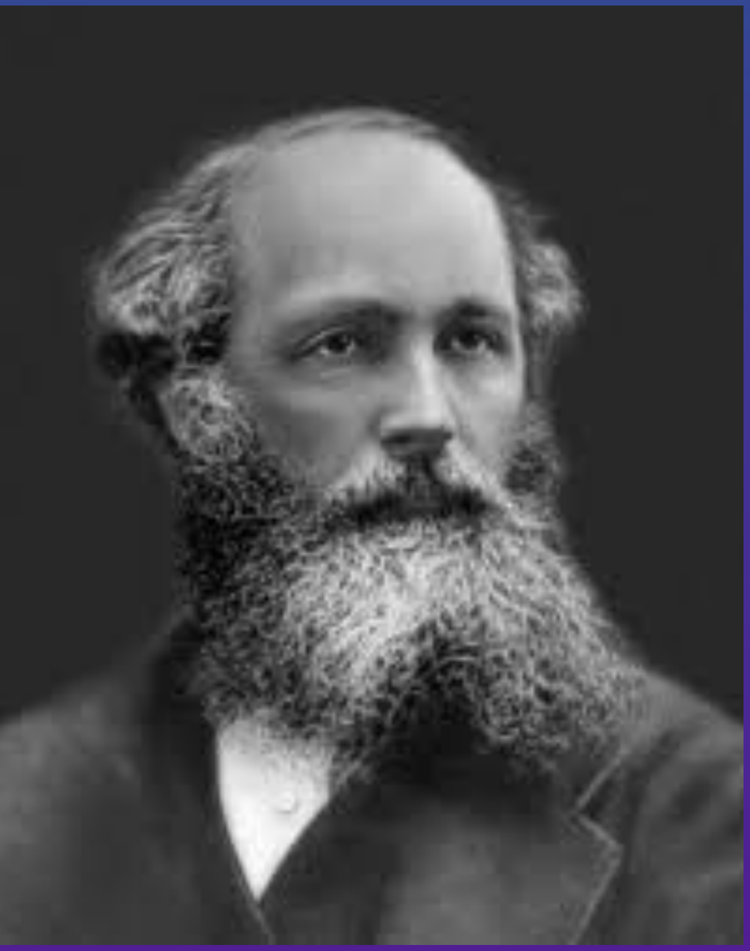

Peter Tait

**Not interesting at all.**



Kelvin

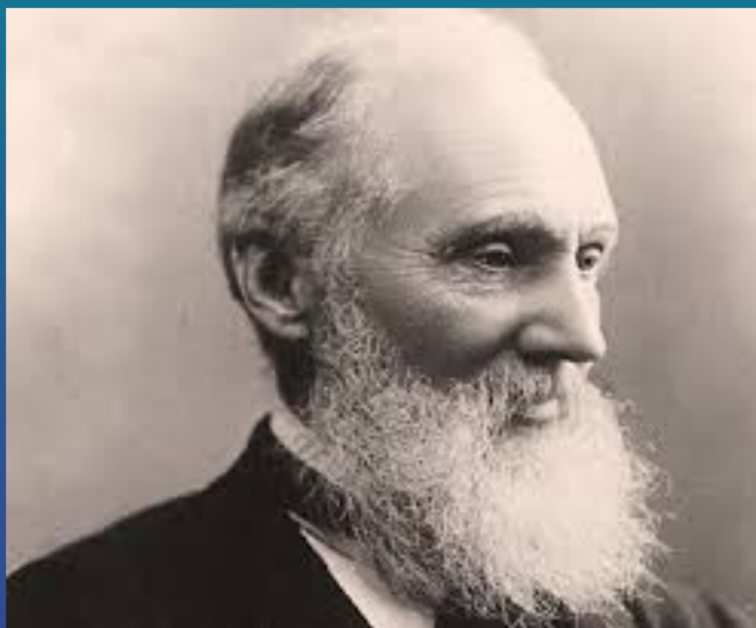
**Very interesting.**



Maxwell

$H_2$

# After 10 Years!

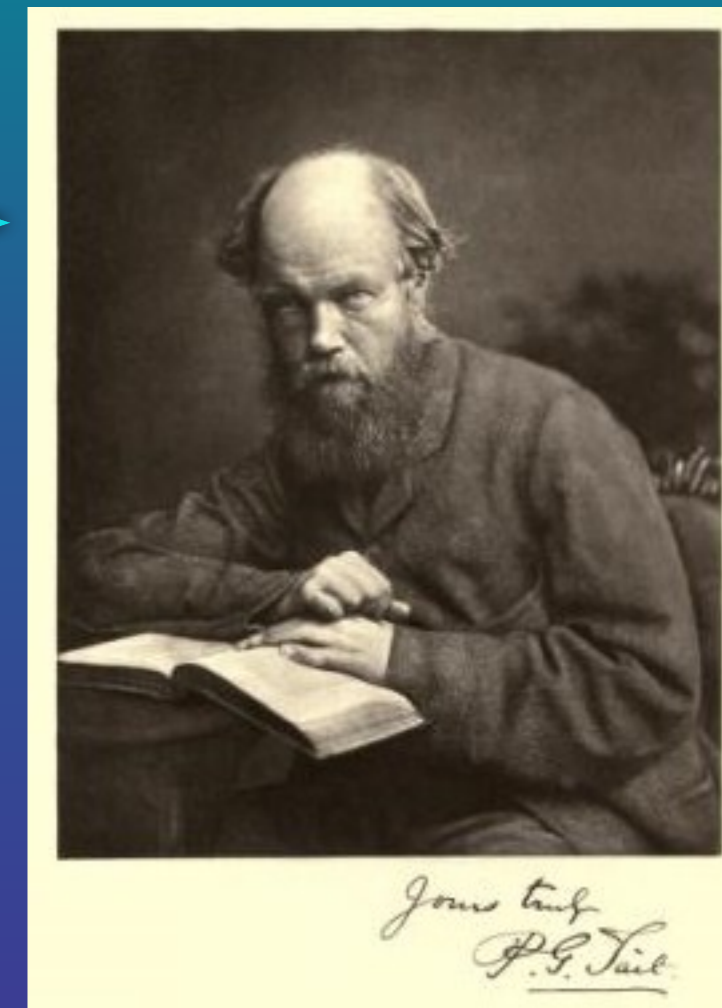


Kelvin

Very interesting.

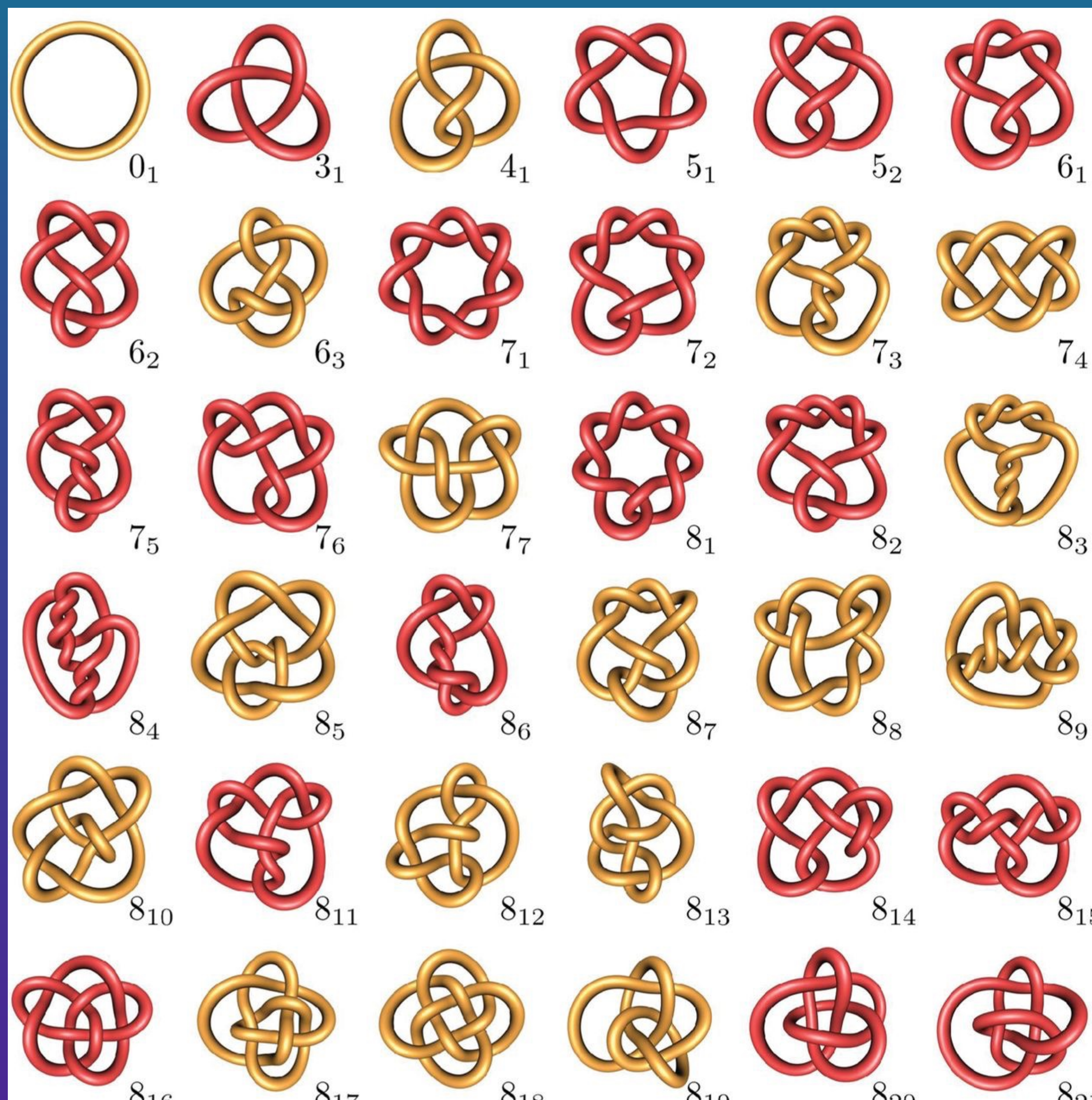
Not interesting at all.

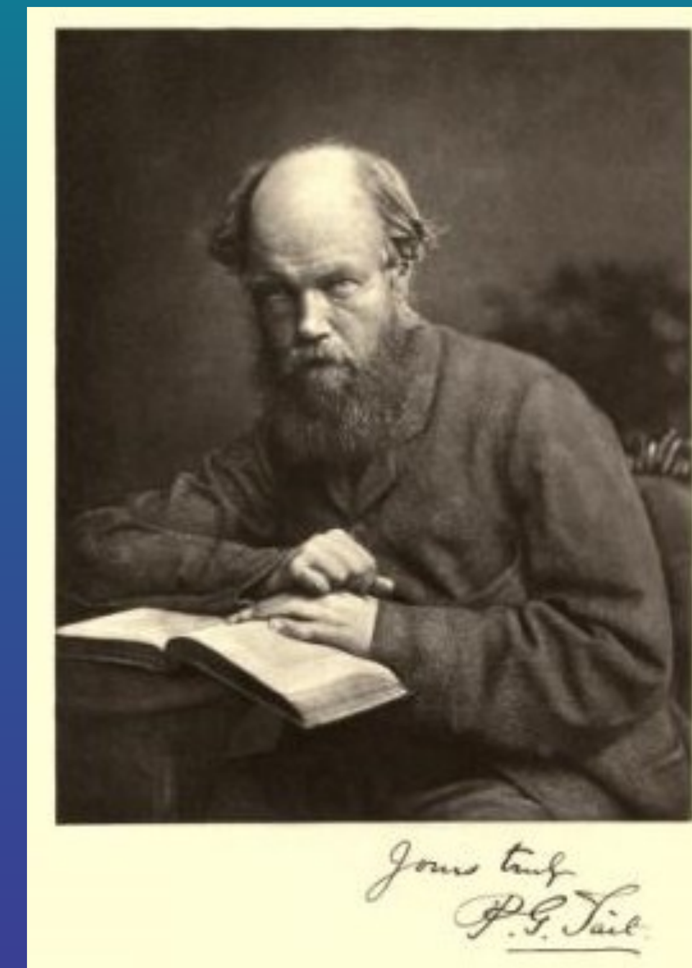
Maxwell

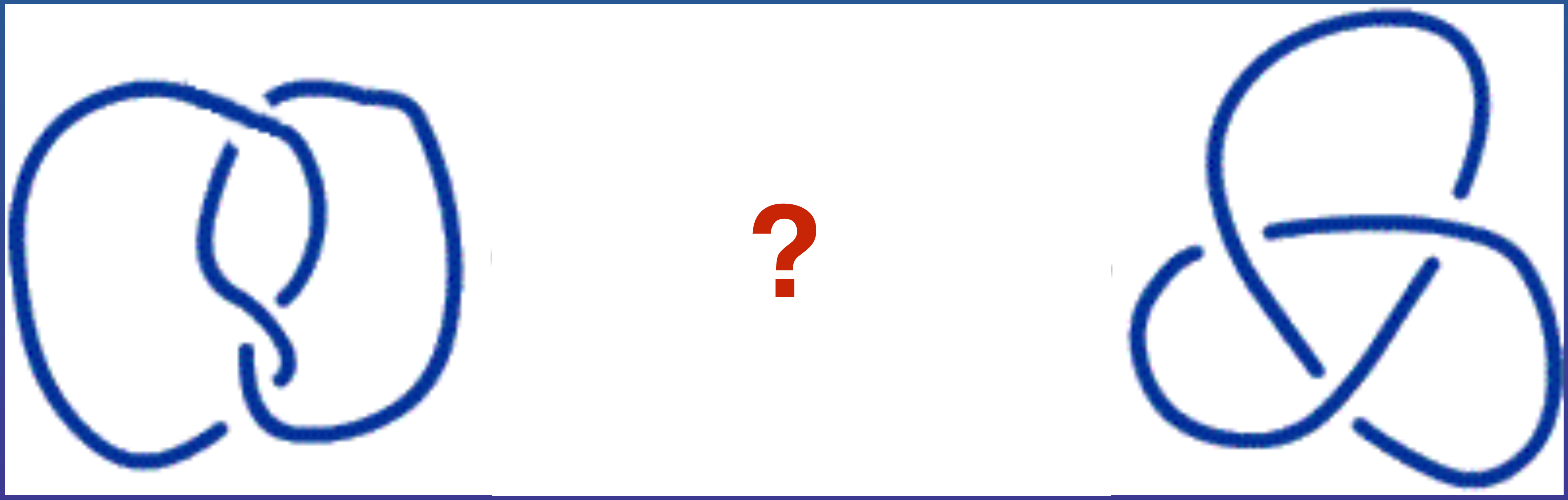


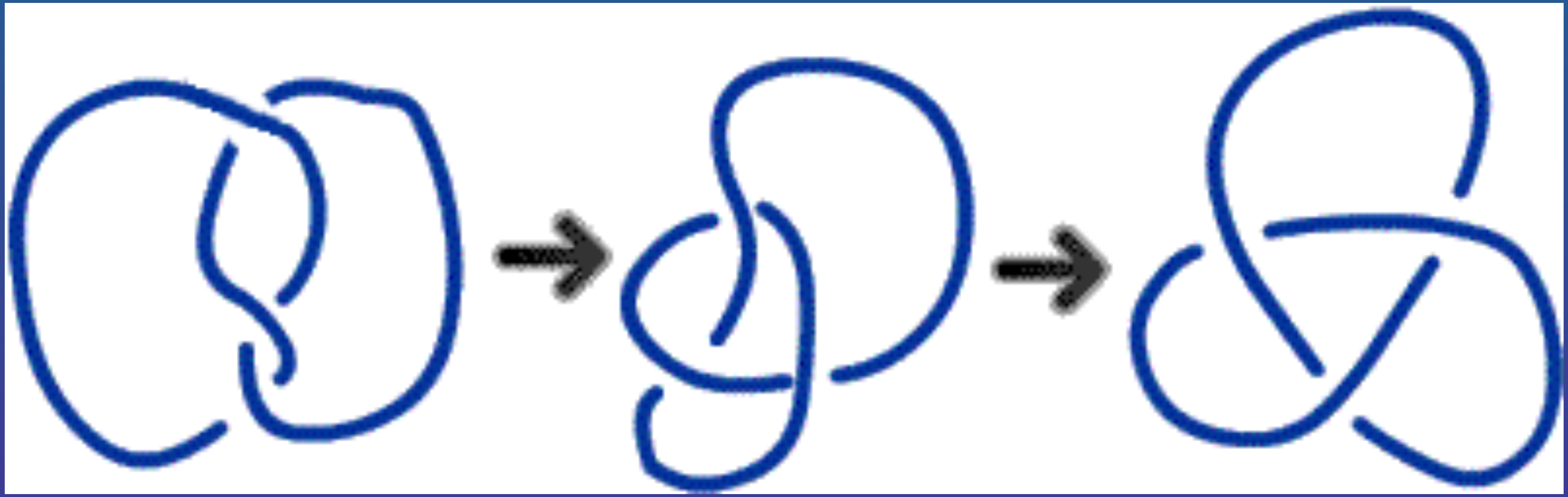
$H_2$

# Knots and periodic table of elements



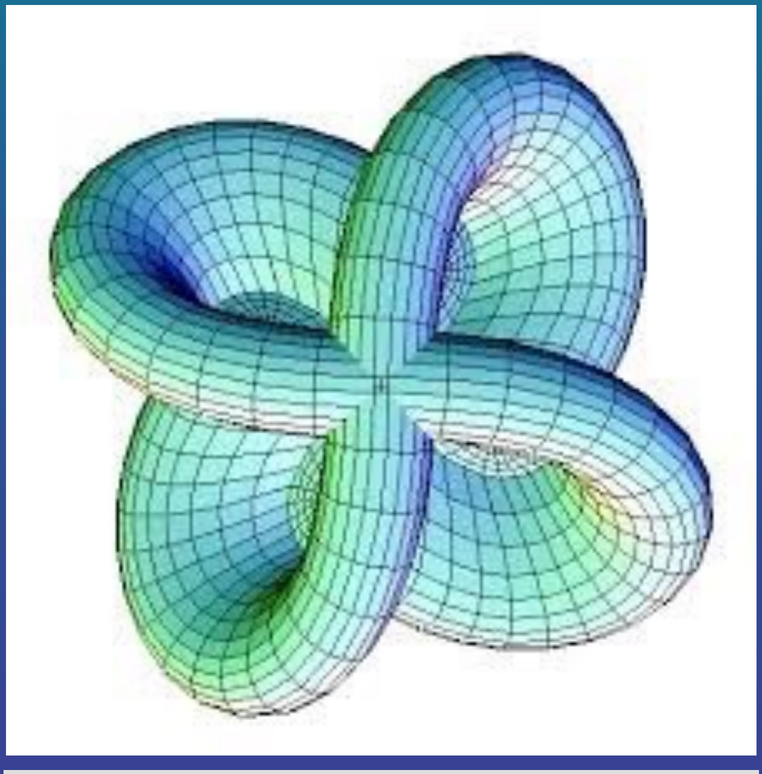



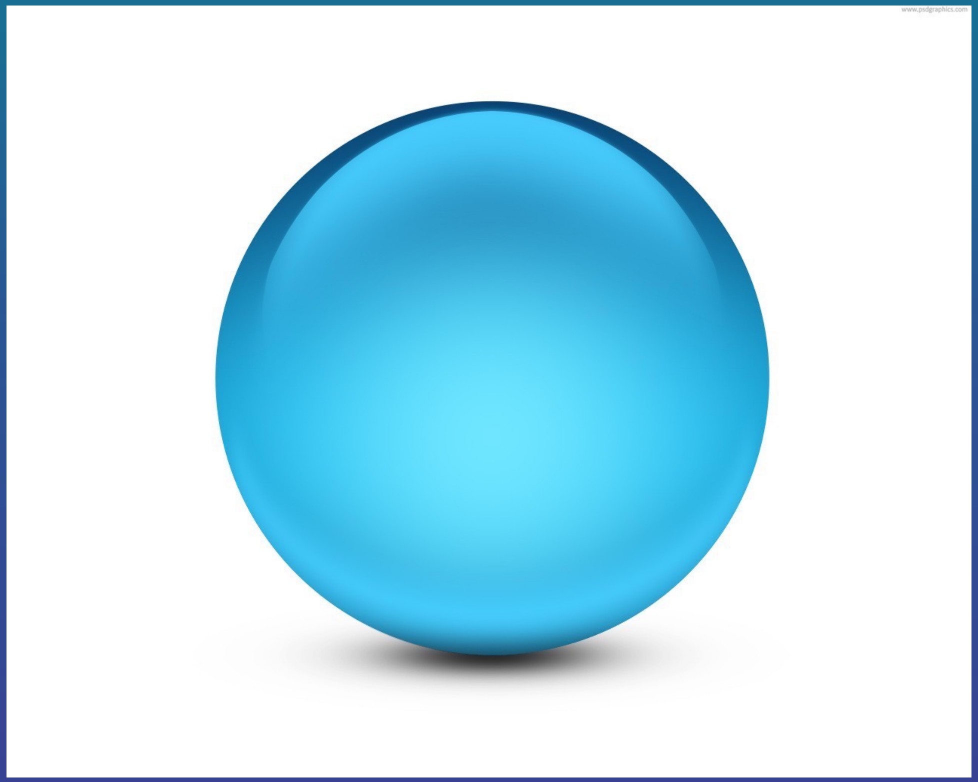


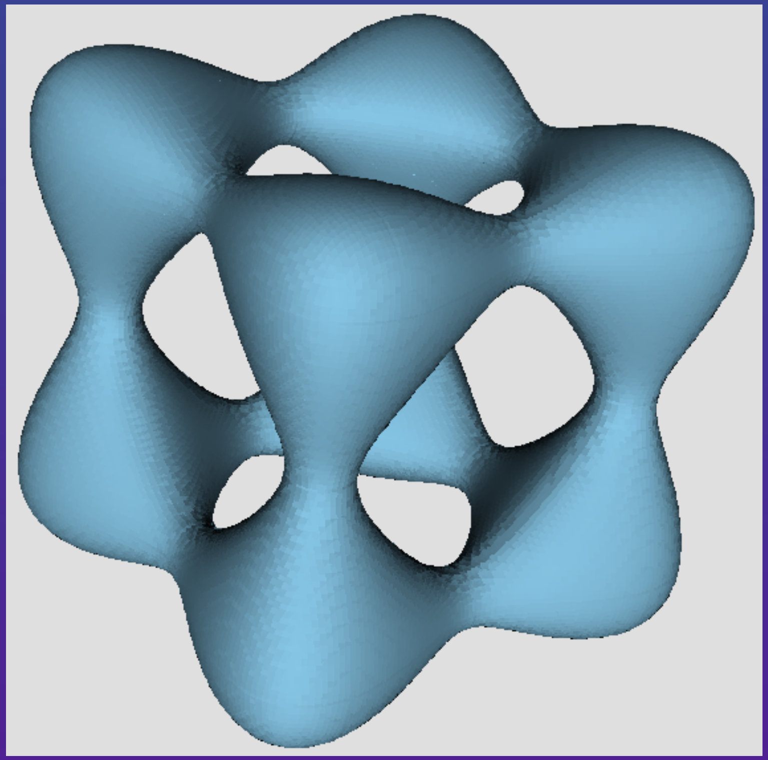


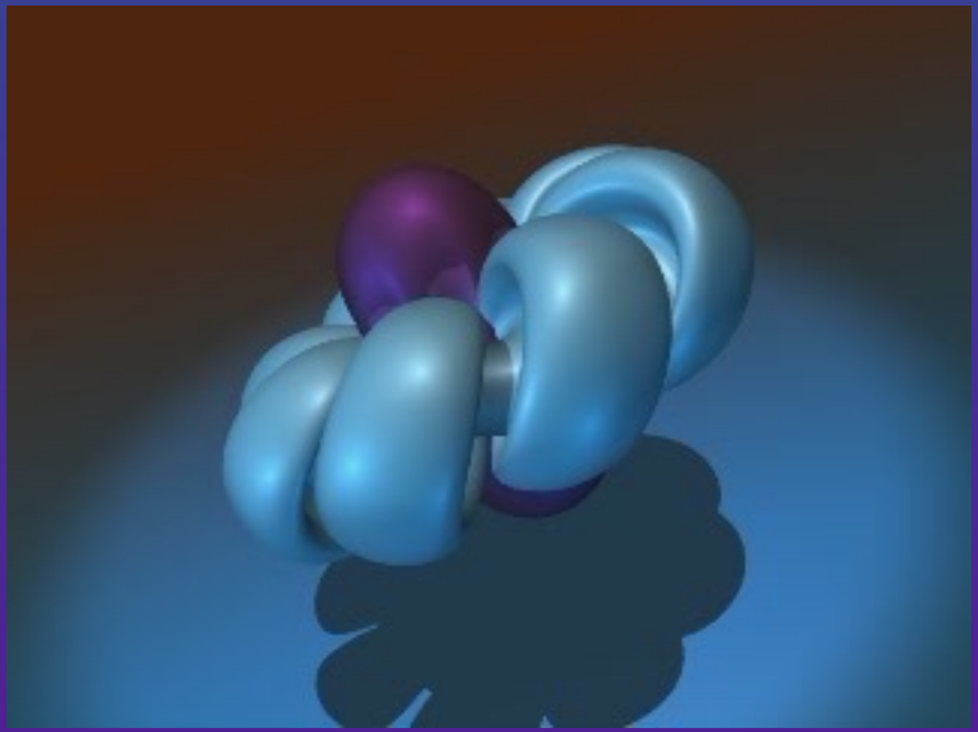
# Two Dimensional Manifolds

# Three Dimensional Manifolds?



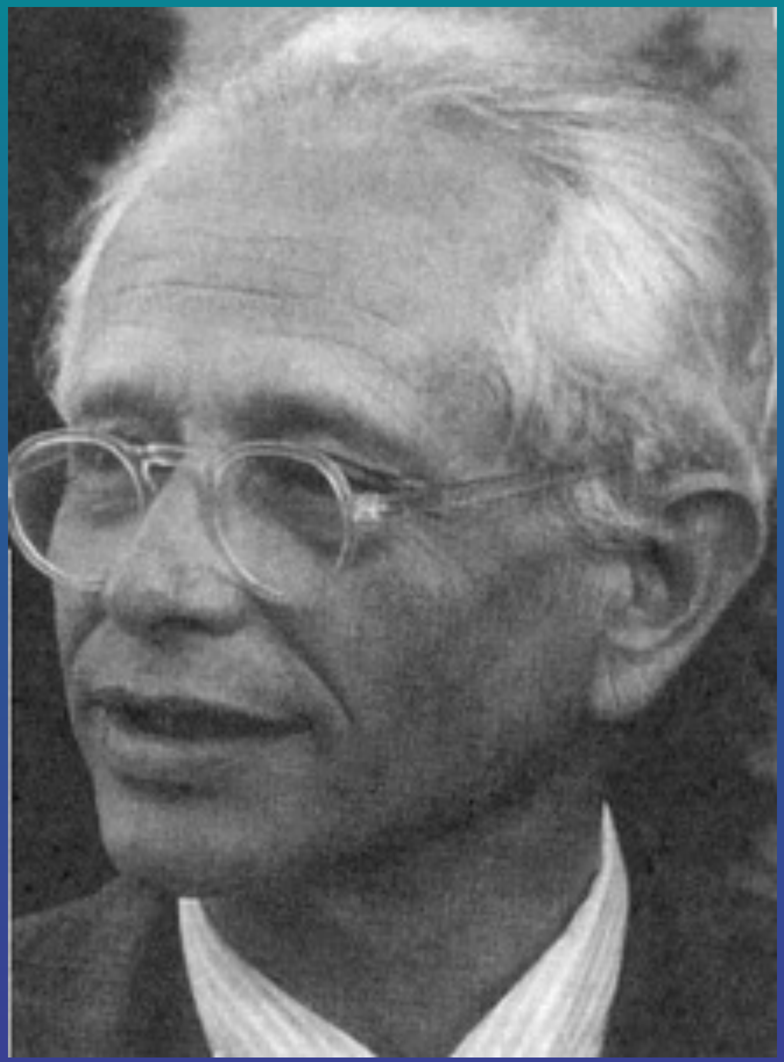






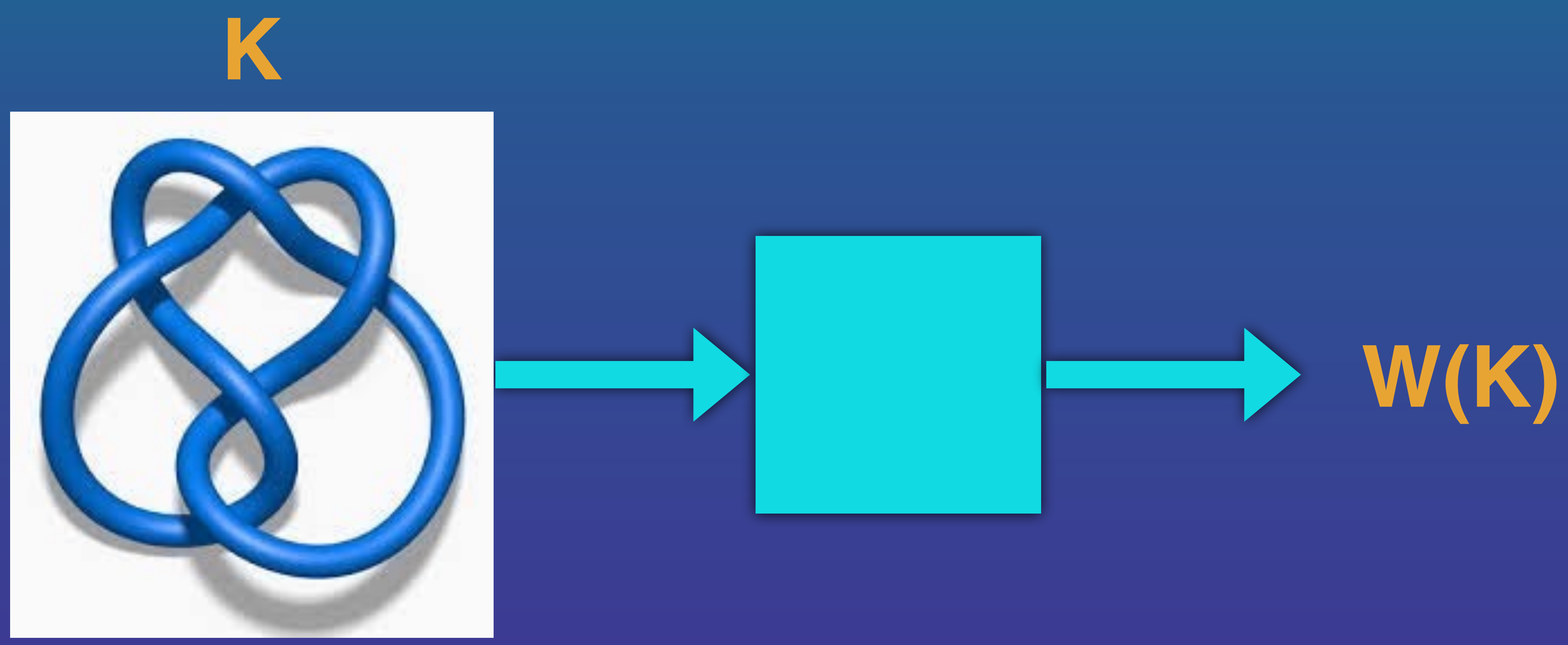
# Dehn Surgery

$S_3 \longrightarrow M$

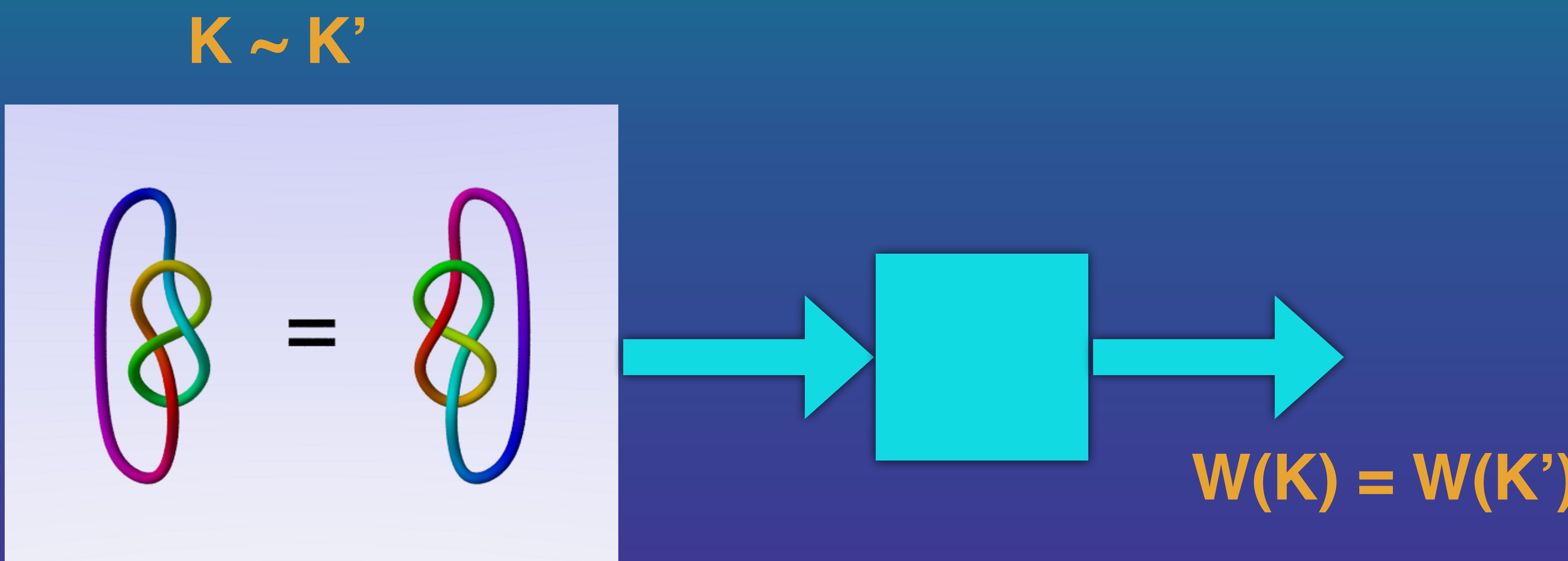


Max Dehn

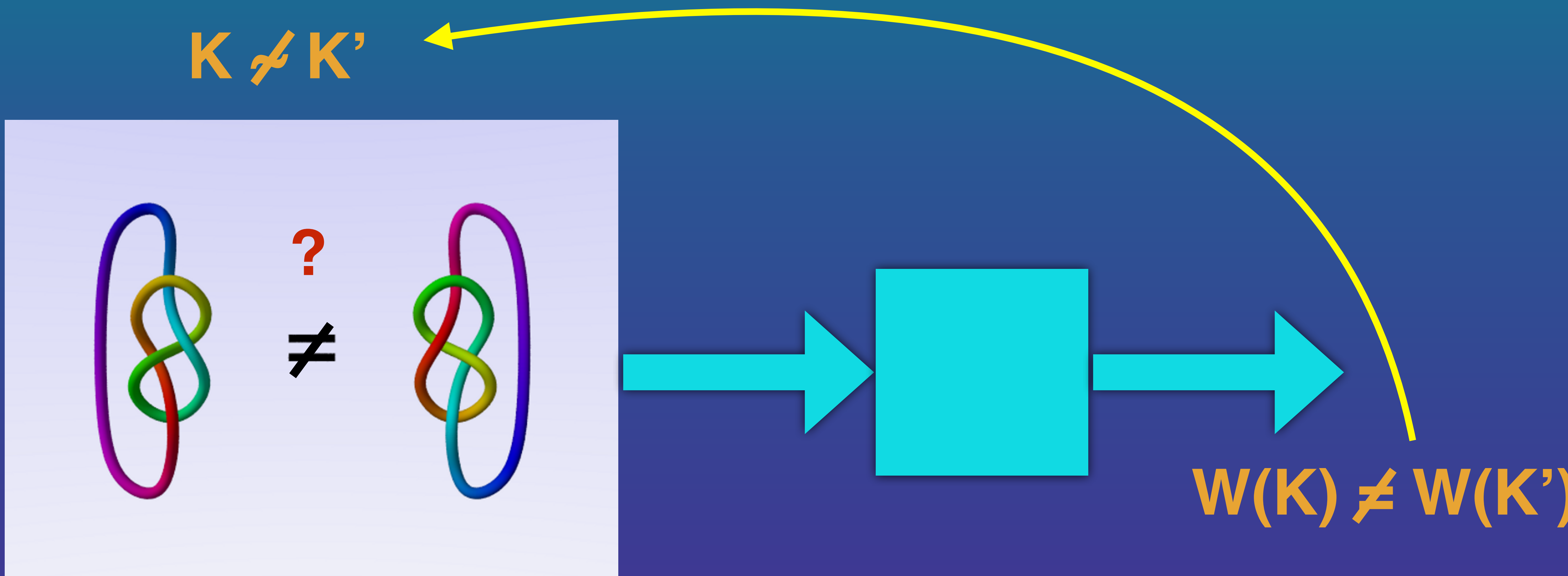
# Knot Invariants



# Knot Invariants



# Knot Invariants



Jones Polynomial

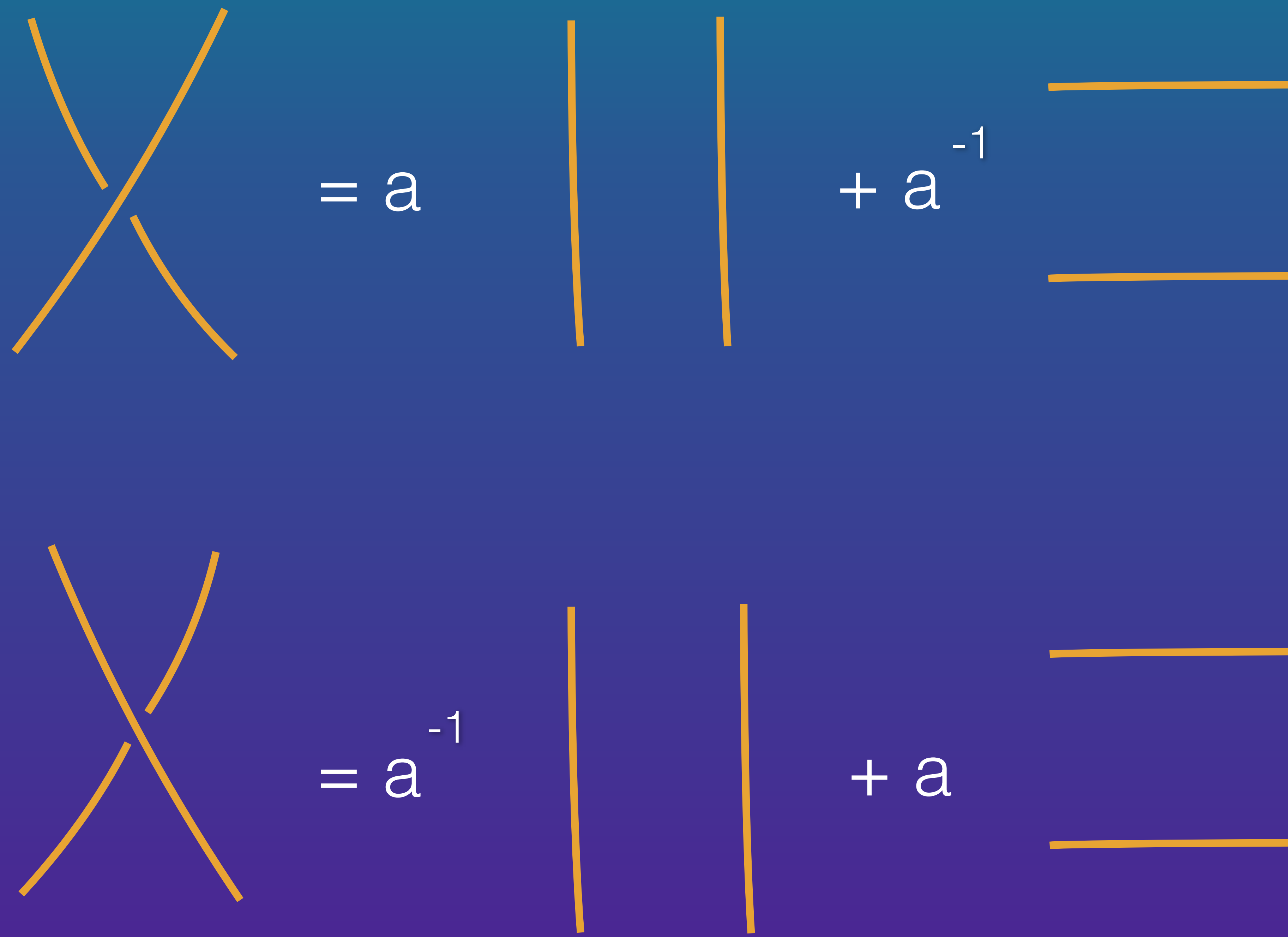
Kauffman Polynomial

# Kauffman Bracket ~ Jones Polynomial

 $= a$  $+ a^{-1}$ 

 $= - a^2 - a^{-2}$

Kauffman Bracket ~ Jones Polynomial



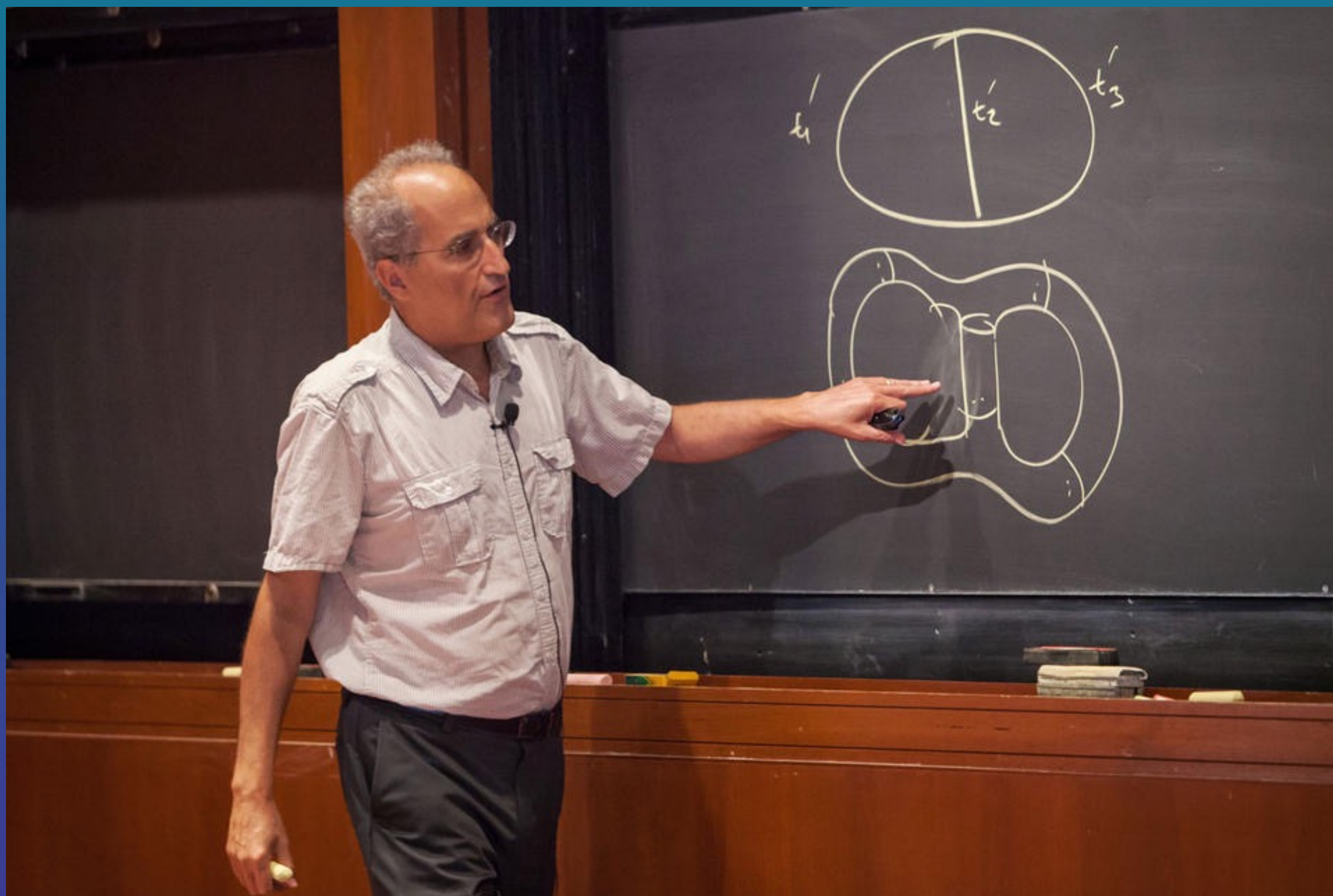

Opening each crossing → Two simpler diagrams

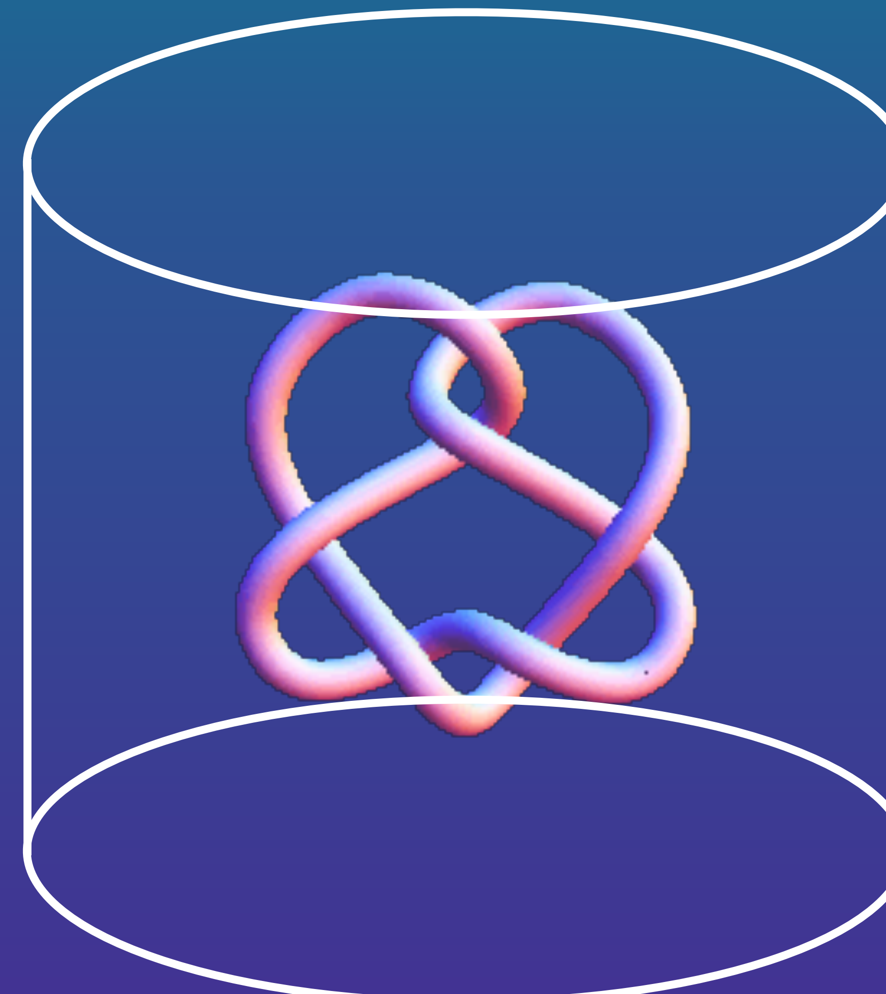
# A computationally Hard Problem

$2^N$ Crossings

# Quantum Field Theory and the Jones Polynomial

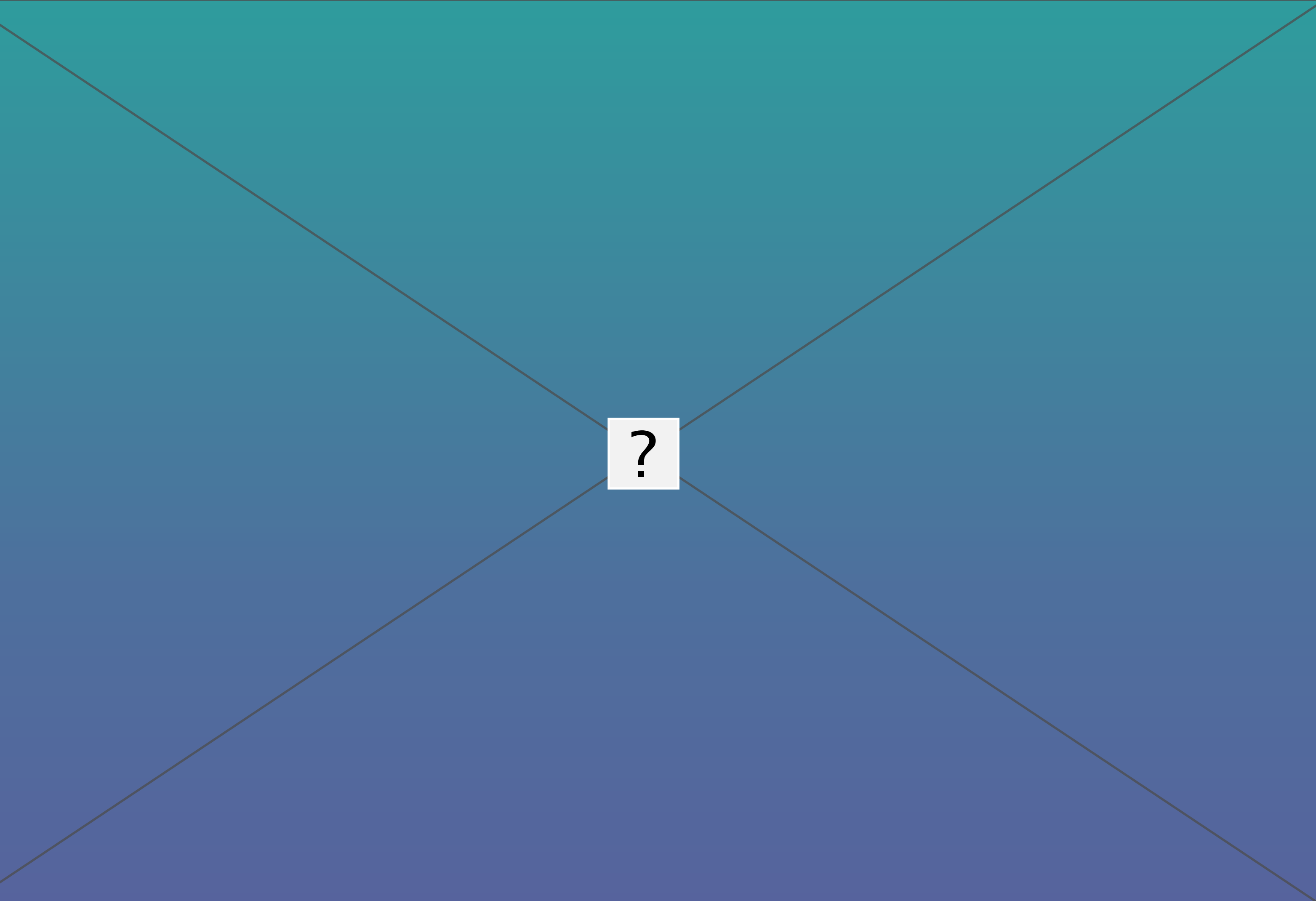
$$Z = \int dA e^{iS}$$





$$\langle W_K \rangle = \int dA W_K[A] e^{iS}$$

$$S = \int d^3x \; Tr\left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right]$$



Michael Freedman, Microsoft Station Q
University of California, Santa Barbara

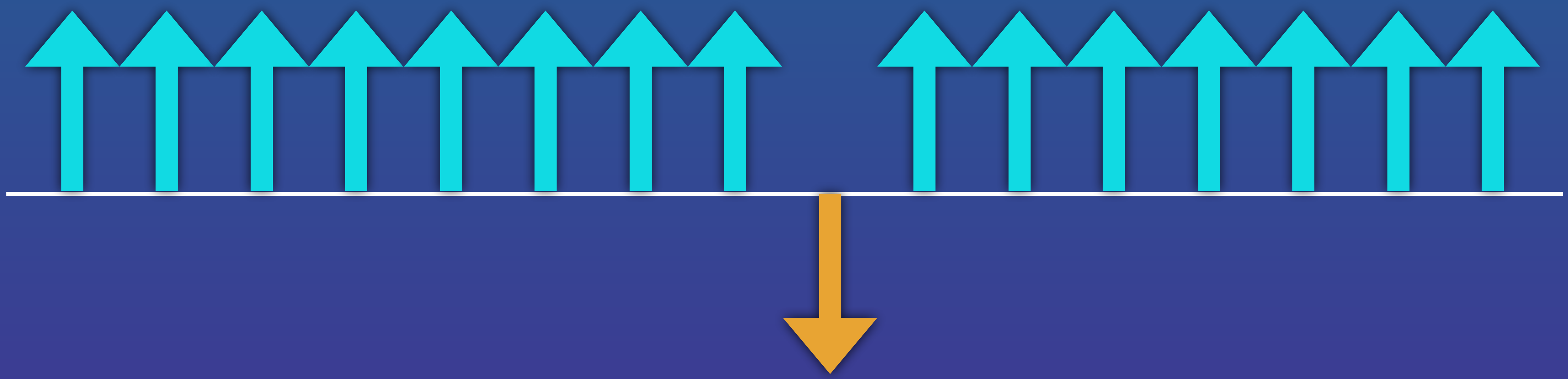
Alexi Kitaev,
Caltech

# Topological Quantum Computation

# Error Free Quantum Computation

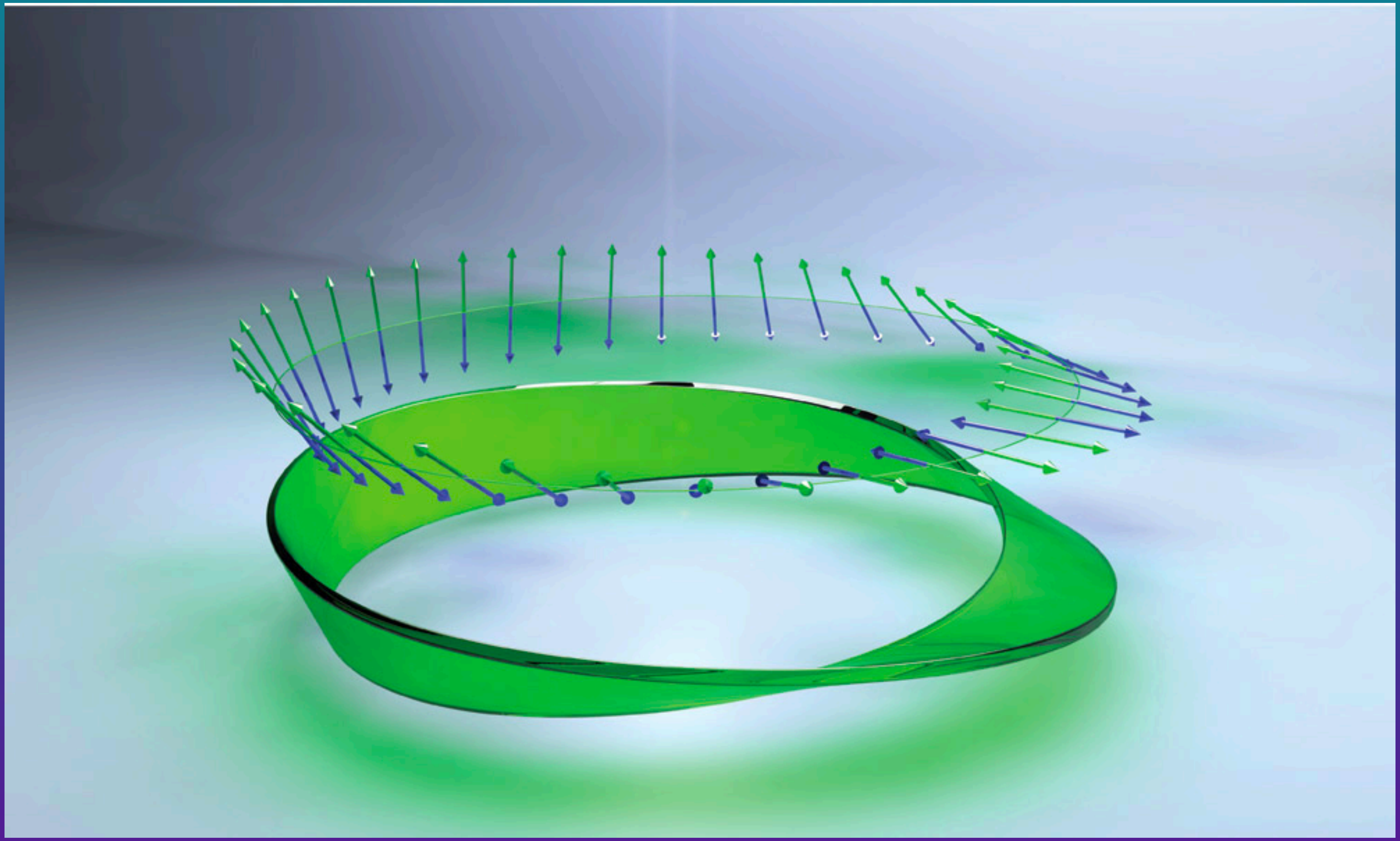
# An excitation ( a quasi particle) which is not topological

Spins in Ising Model



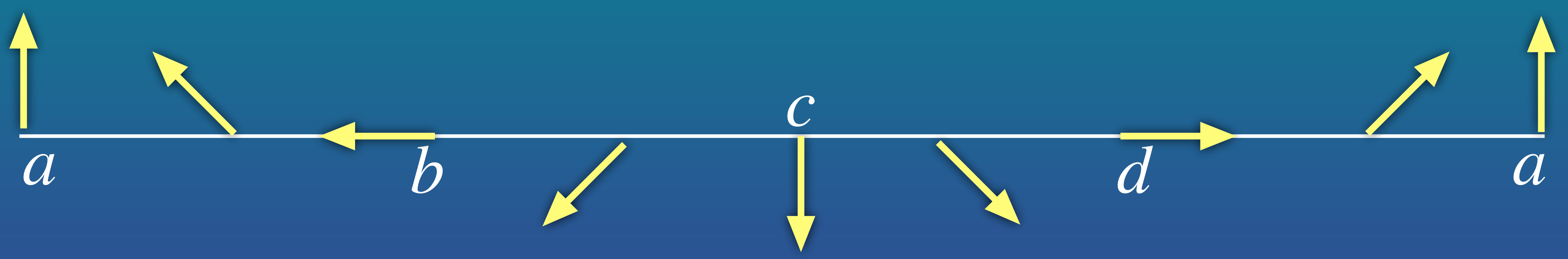
You flip the spin by a local operator and the excitation is removed!

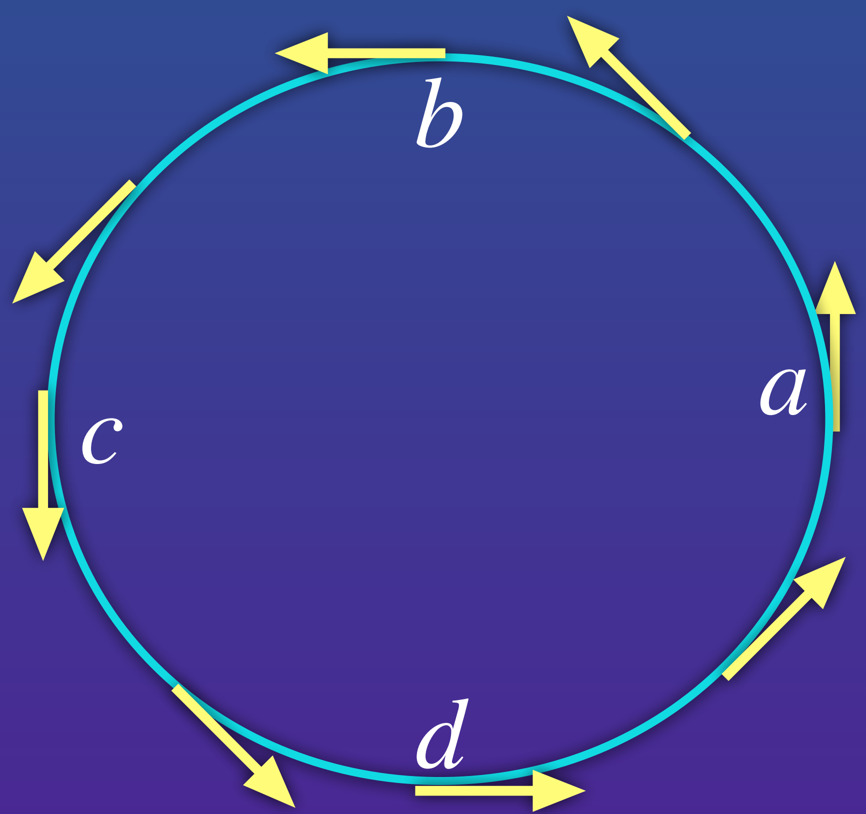
# An excitation ( a quasi particle) which is topological



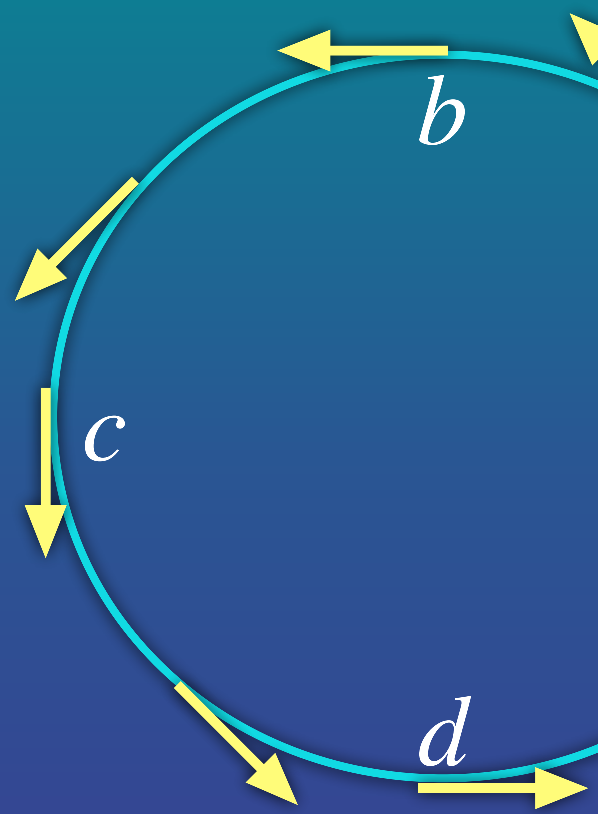
You cannot remove the excitation by local operations.flip the spin by a local operator and the excitation is removed!

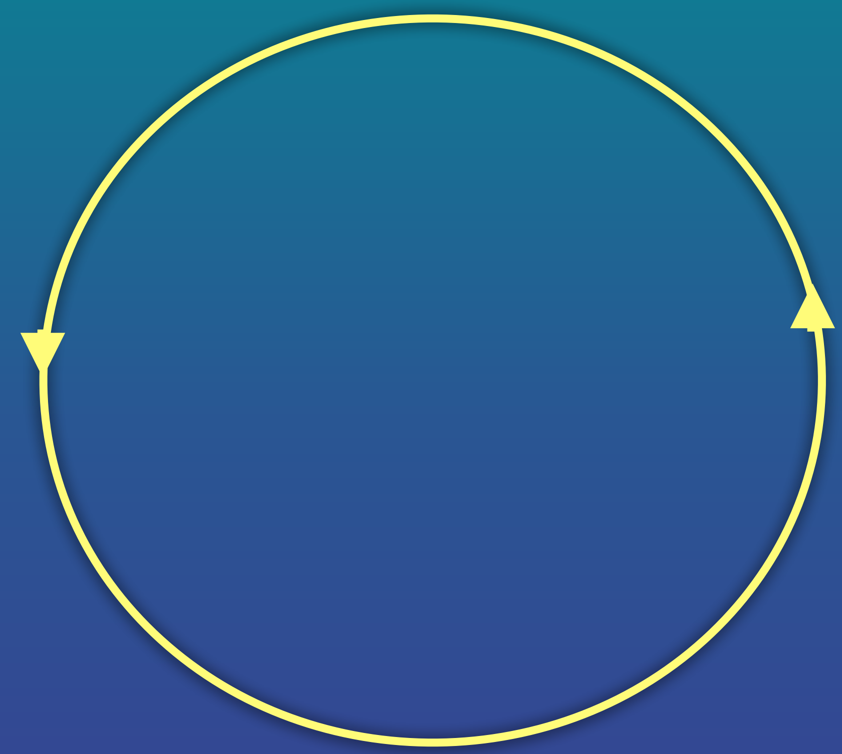
An excitation ( a quasi-particle) which is topological.




Topology protects the excitation.



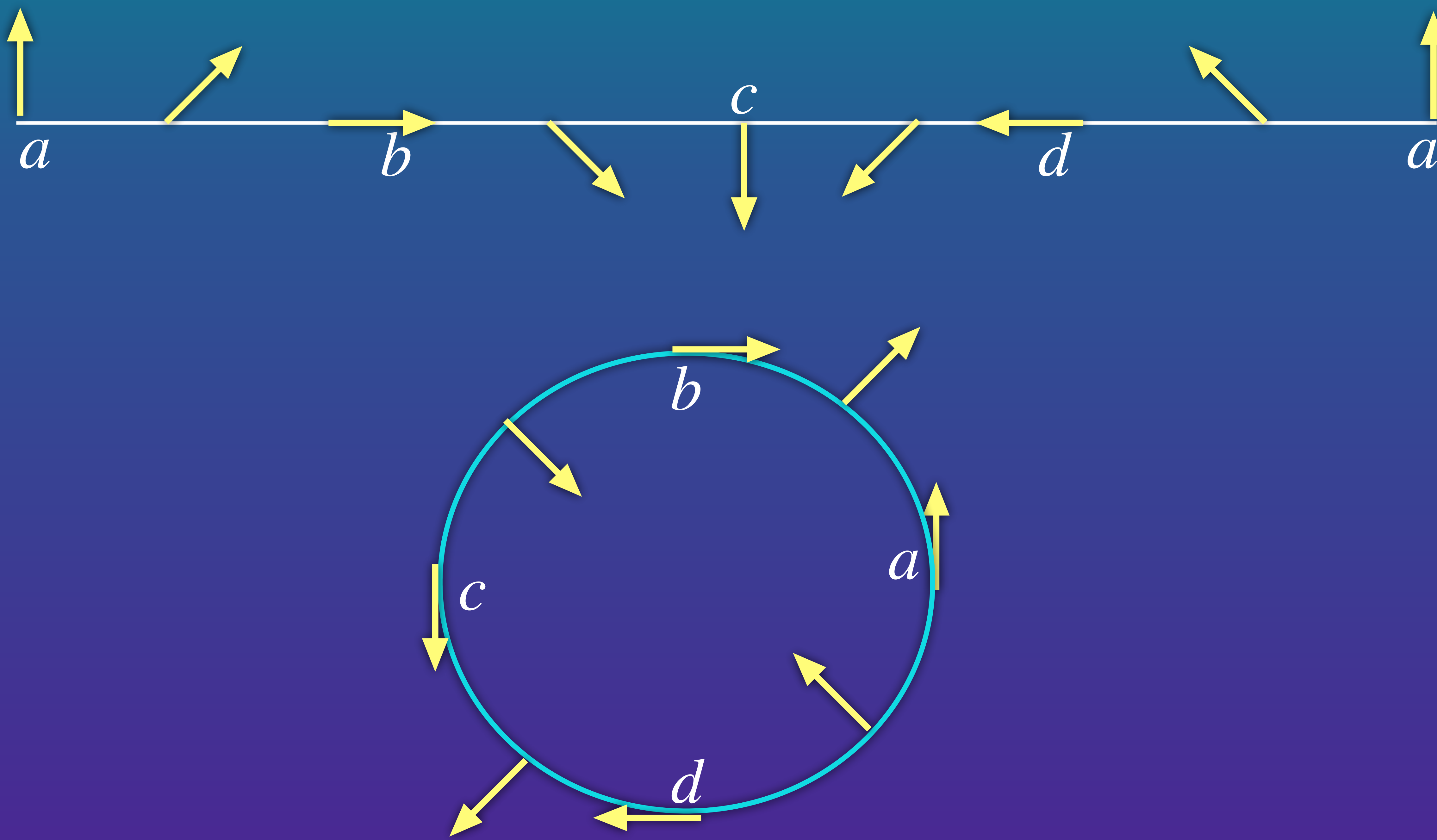



$$\phi : S^1 \longrightarrow S^1$$

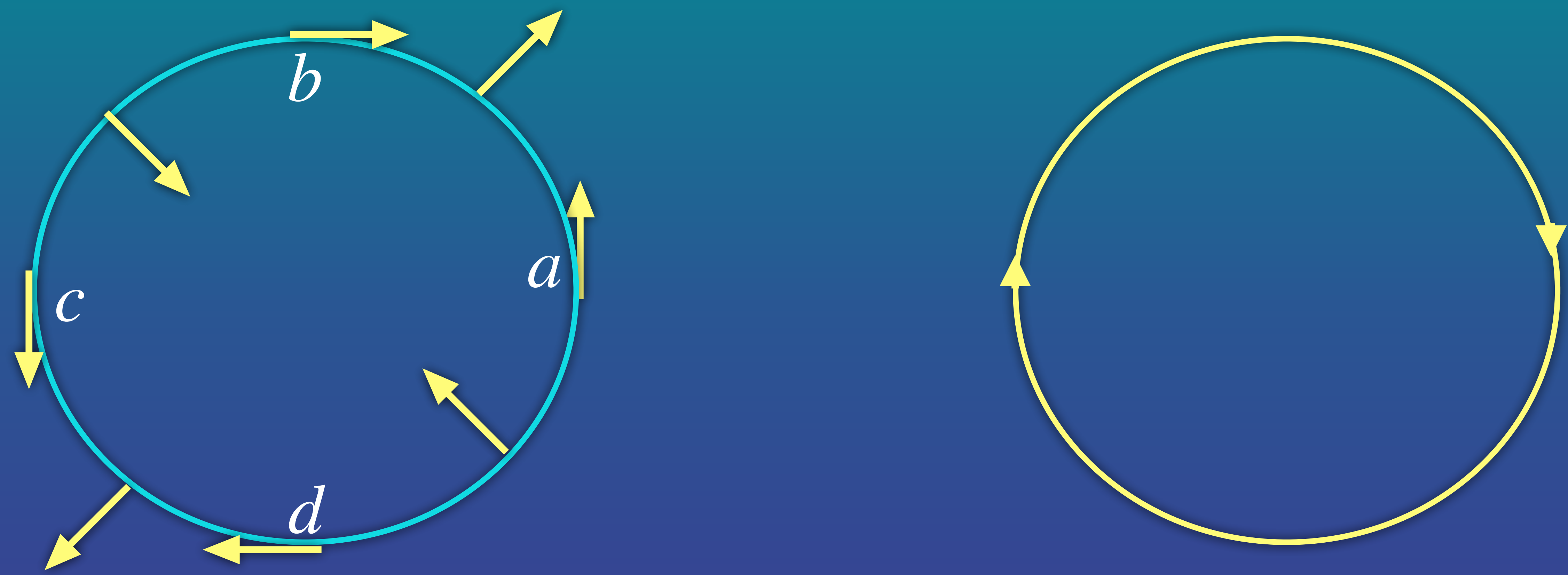
$$\phi : \text{Space} \longrightarrow \text{Spin}$$

Winding number = q = 1

An excitation ( a quasi-particle) which is topological.




Topology protects the excitation.

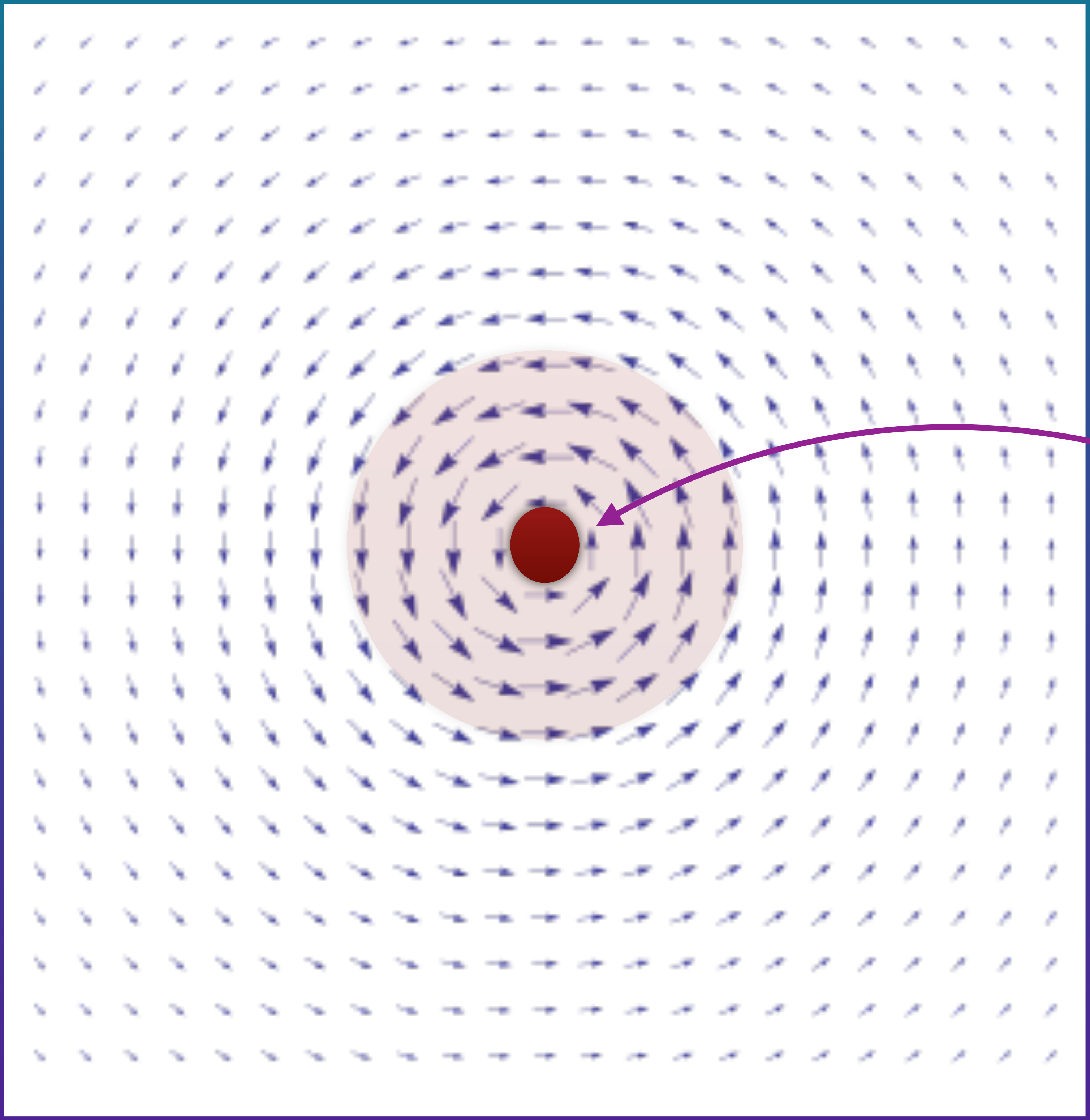



$$\phi : S^1 \longrightarrow S^1$$

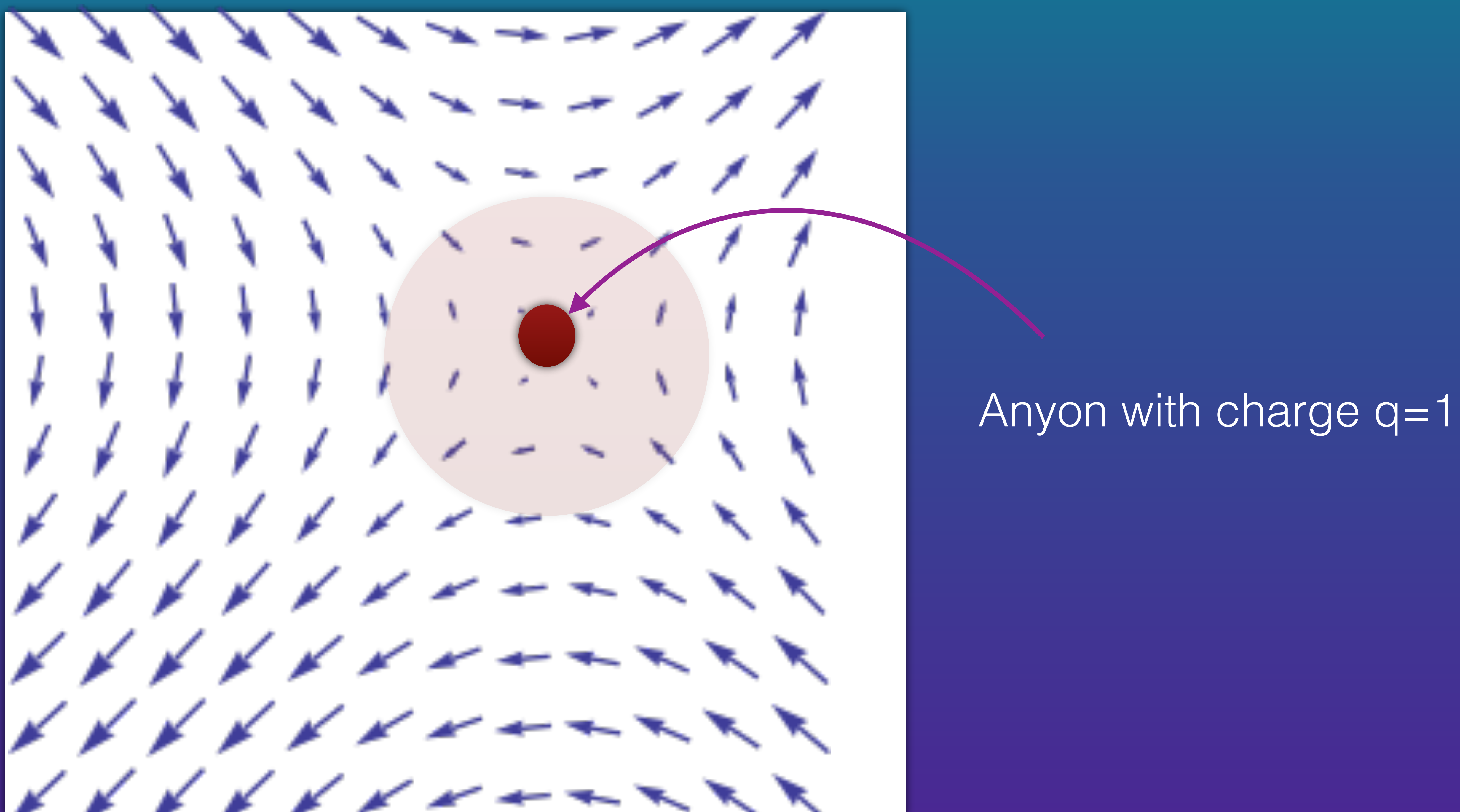
$$\phi : \text{Space} \longrightarrow \text{Spin}$$

Winding number = q = -1

# Quasi particles have charge!



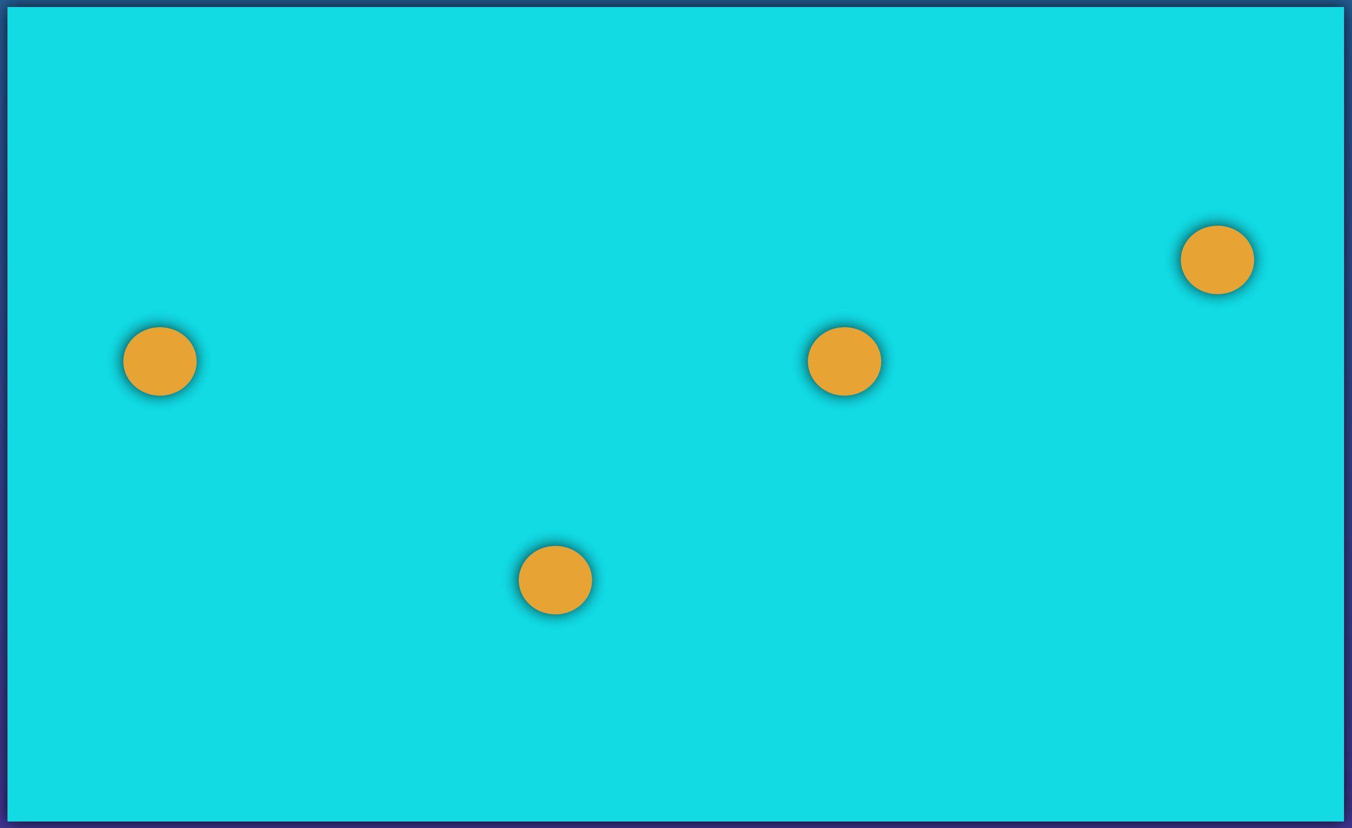
Anyon with charge q=1


Anyon with charge q=1

# The basic principles

1- There are many body systems whose ground states have topological charge

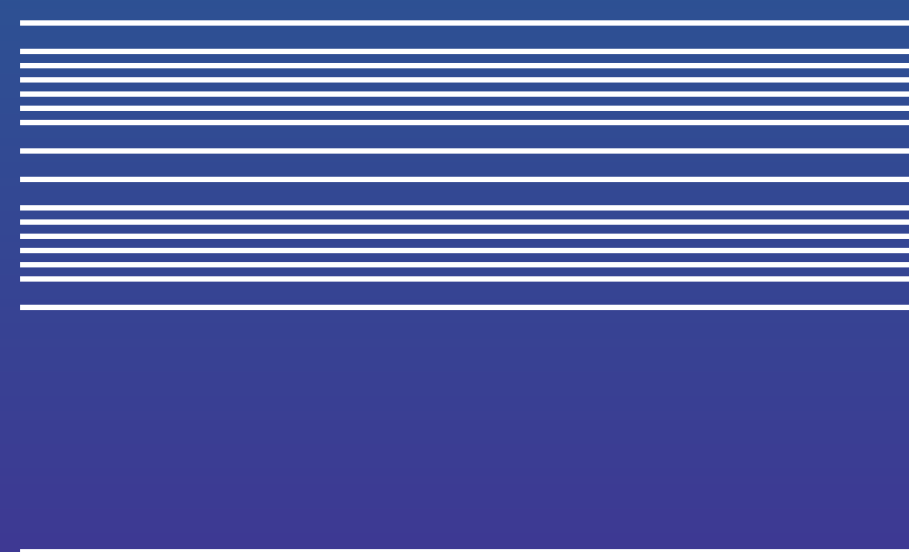
2-The degeneracy of the ground state depends on the number of these particles.



The higher the number of charges → The higher the degeneracy

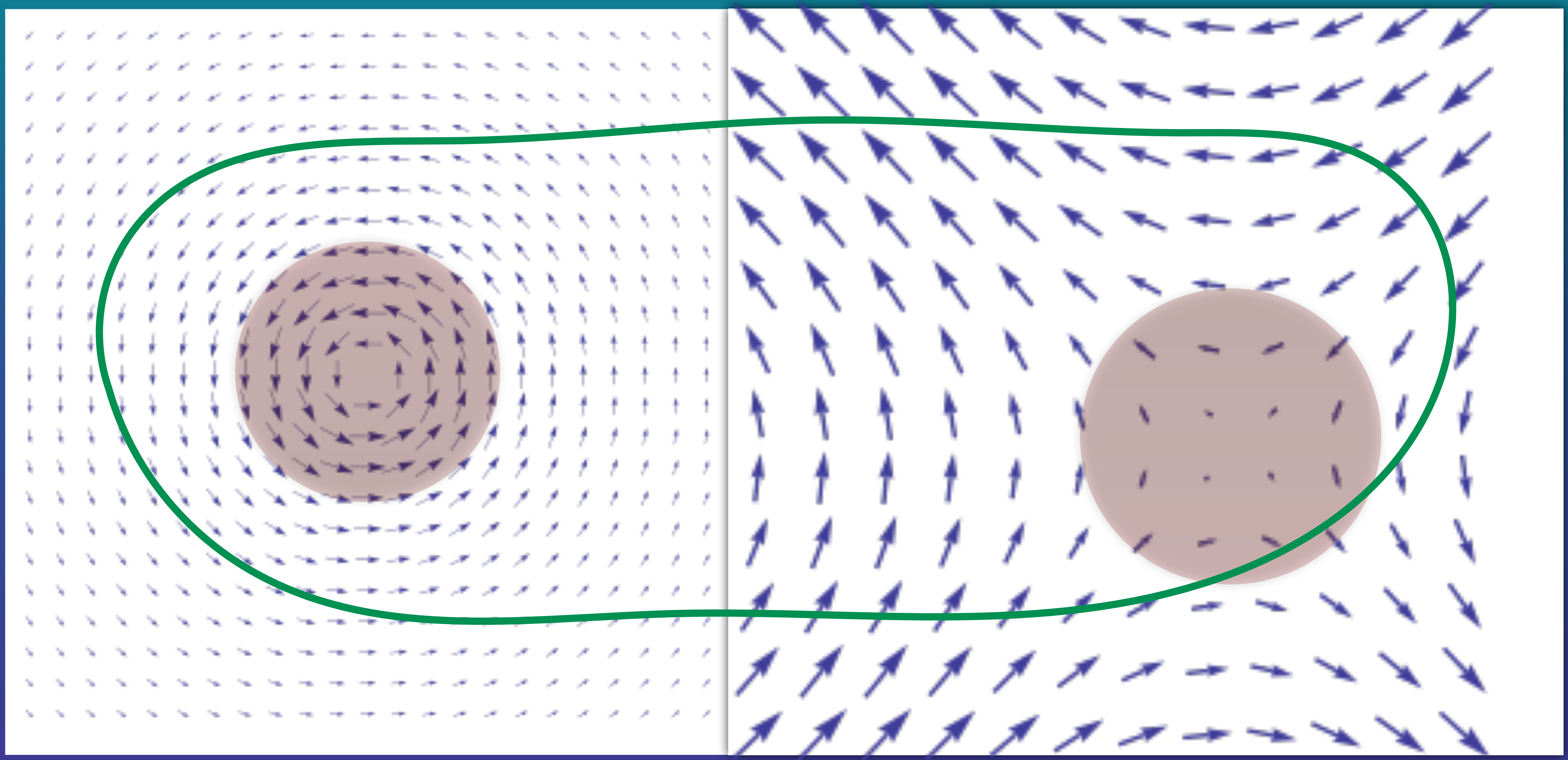
$$|g_1\rangle, |g_2\rangle, |g_3\rangle, \ldots\ldots |g_N\rangle$$

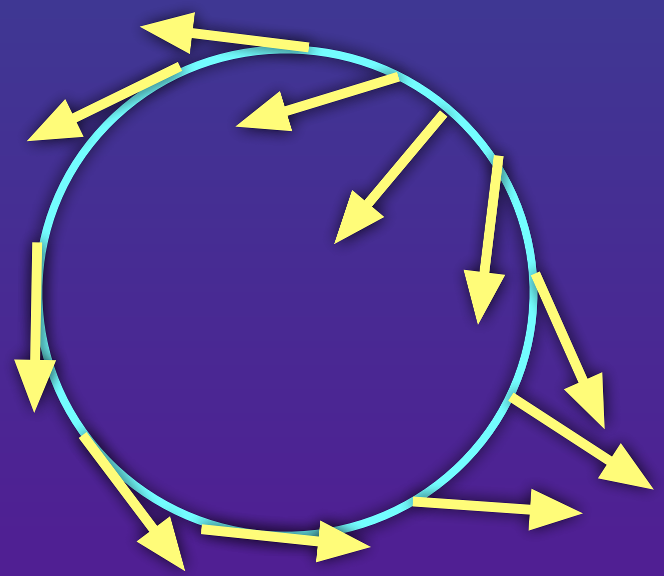
# 3-The system has a gap.



We always stay in the ground space.

# Combination of topological charges

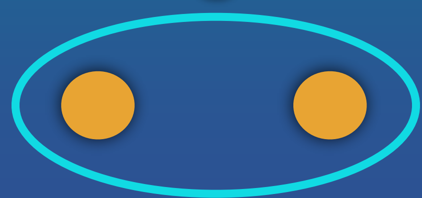




$$q = 1 + (-1) = 0$$

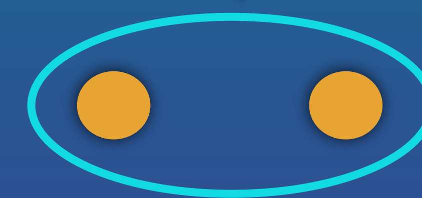
Example: **Fibonacci Anyons** 0 and 1

0



$$1 \times 1 = 0$$

1

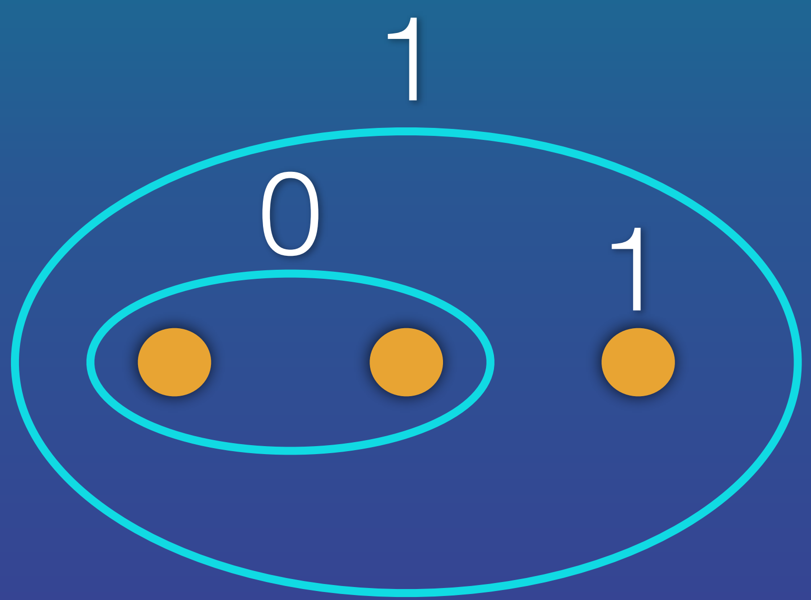


$$1 \times 1 = 1$$

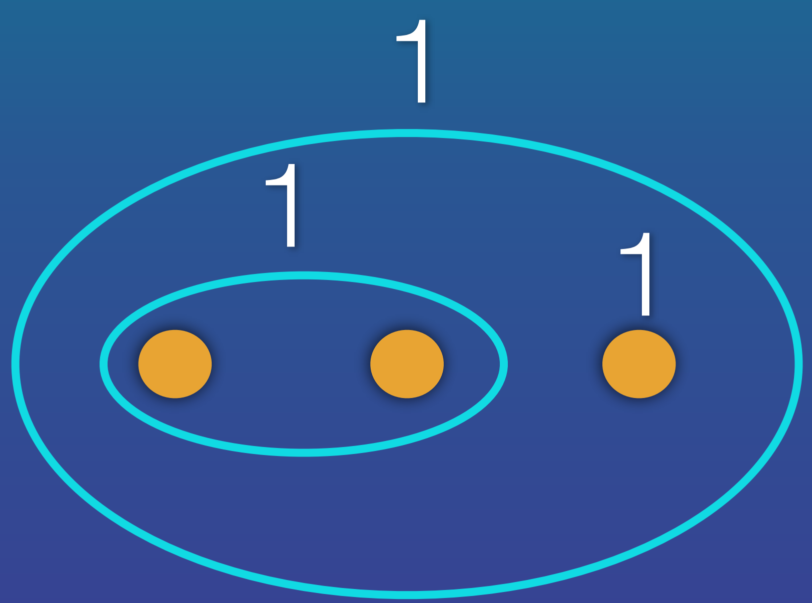
$$1 \times 1 = 0 + 1$$

By two Fibonacci Anyons, we cannot make a qubit since their total charge will be different and so we cannot make linear superposition of them

# By three Fibonacci Anyons, we can make a qubit

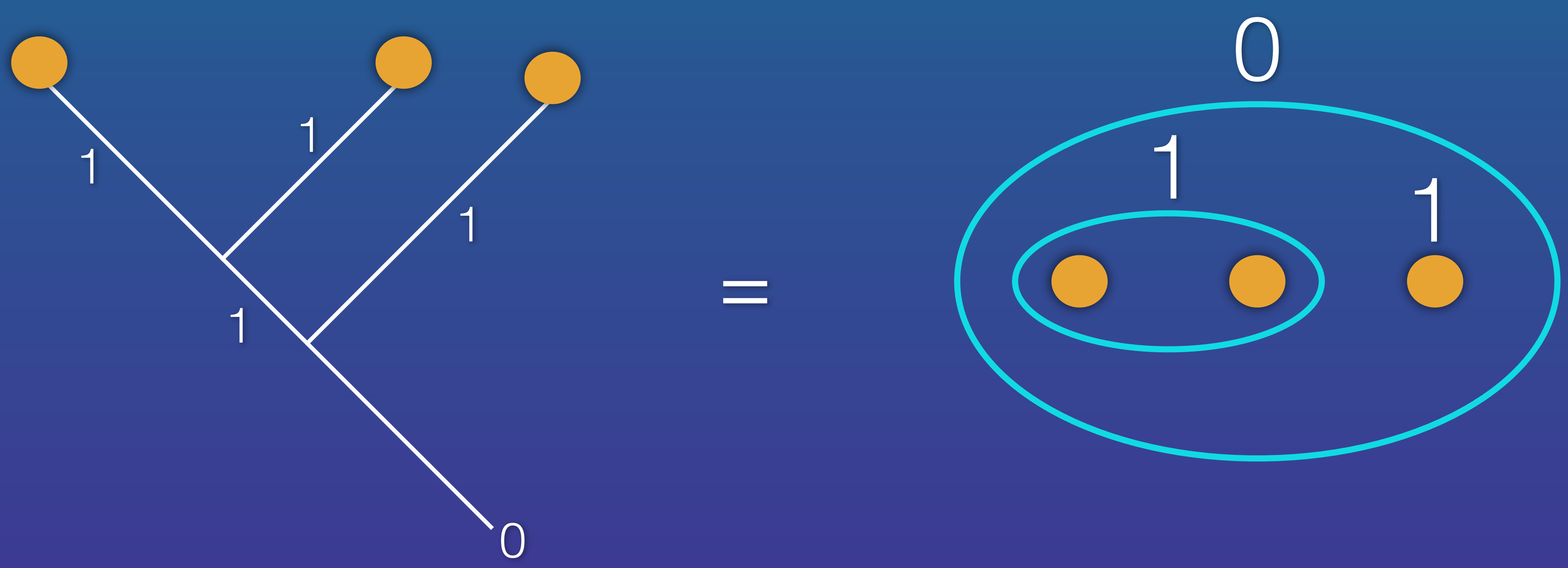



$|0\rangle$



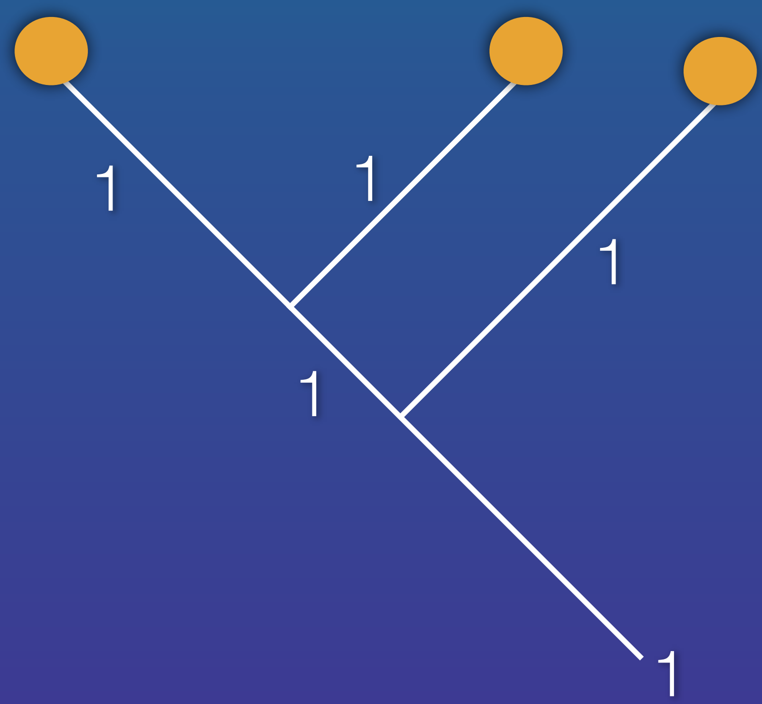

$|1\rangle$

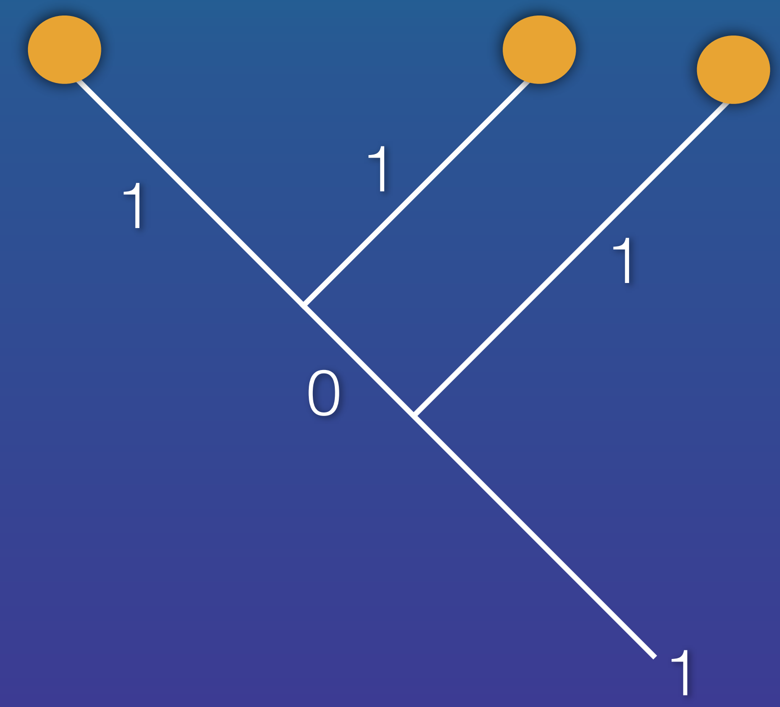
# With three Fibonacci anyons with total charge zero, we cannot make a qubit.




Here is there  is no degeneracy.

With three Fibonacci anyons with total charge one, we can make a qubit.




Here is there  is degeneracy.

The ground state with n Fibonacci anyons with total charge 1, has a degeneracy given by

$$\phi^n$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

$\phi$ = quantum dimension

$$1^n = a_n\ 0 + b_n\ 1$$

$$1^{n+1} = a_n\ 0 \times 1 + b_n\ 1 \times 1$$

$$= a_n\ 1 + b_n\ (0 + 1)$$

$$= b_n\ 0 + (a_n + b_n)\ 1$$

$$a_{n+1} = b_n$$

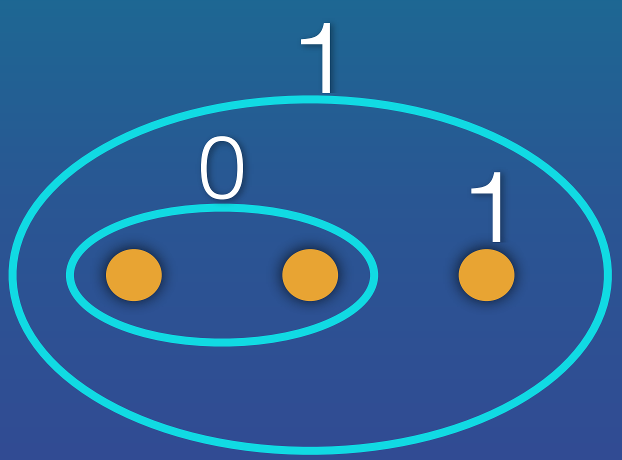
$$b_{n+1} = a_n + b_n$$

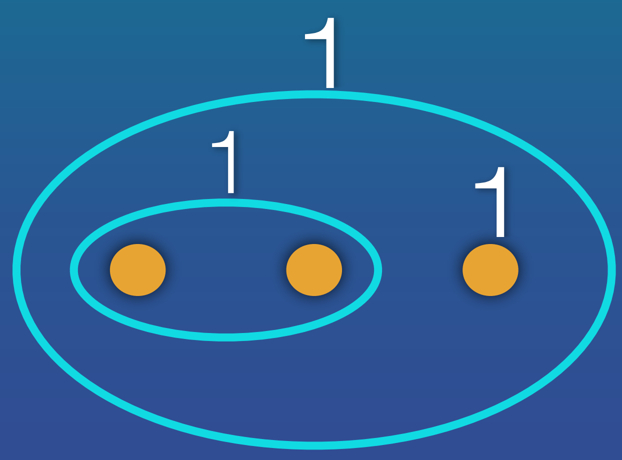
$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\phi_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

$$b_n \sim \phi_+^n$$

# Can we really think of these



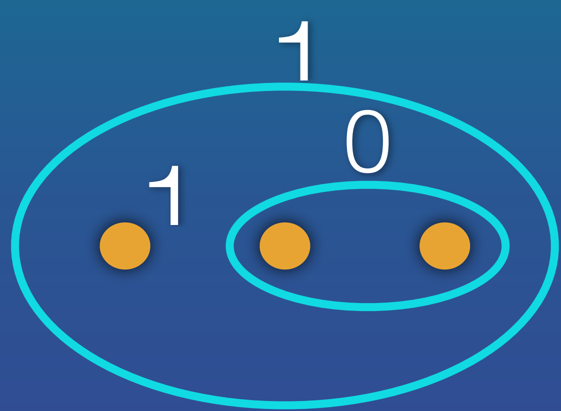



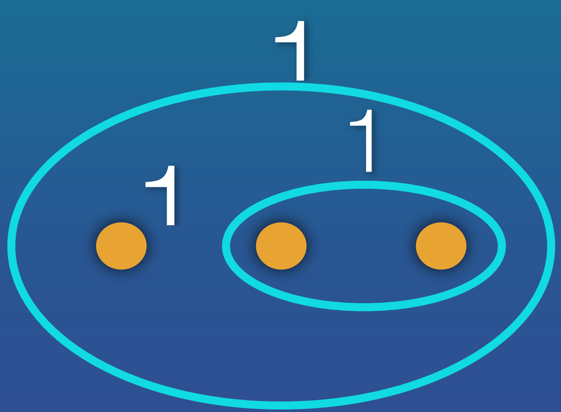

as qubits

$|0\rangle$ and $|1\rangle$

?

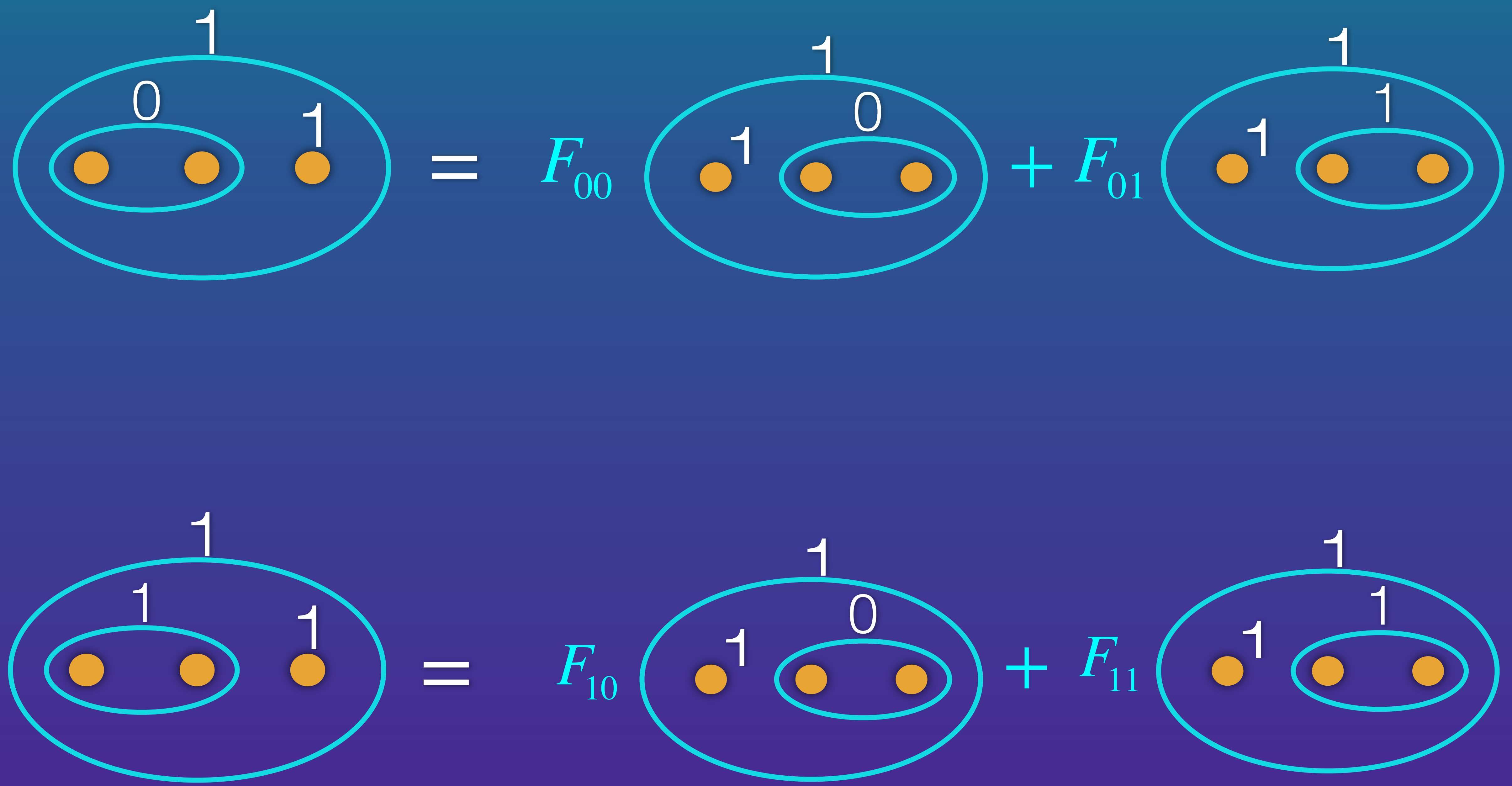
# For example, what are the following?

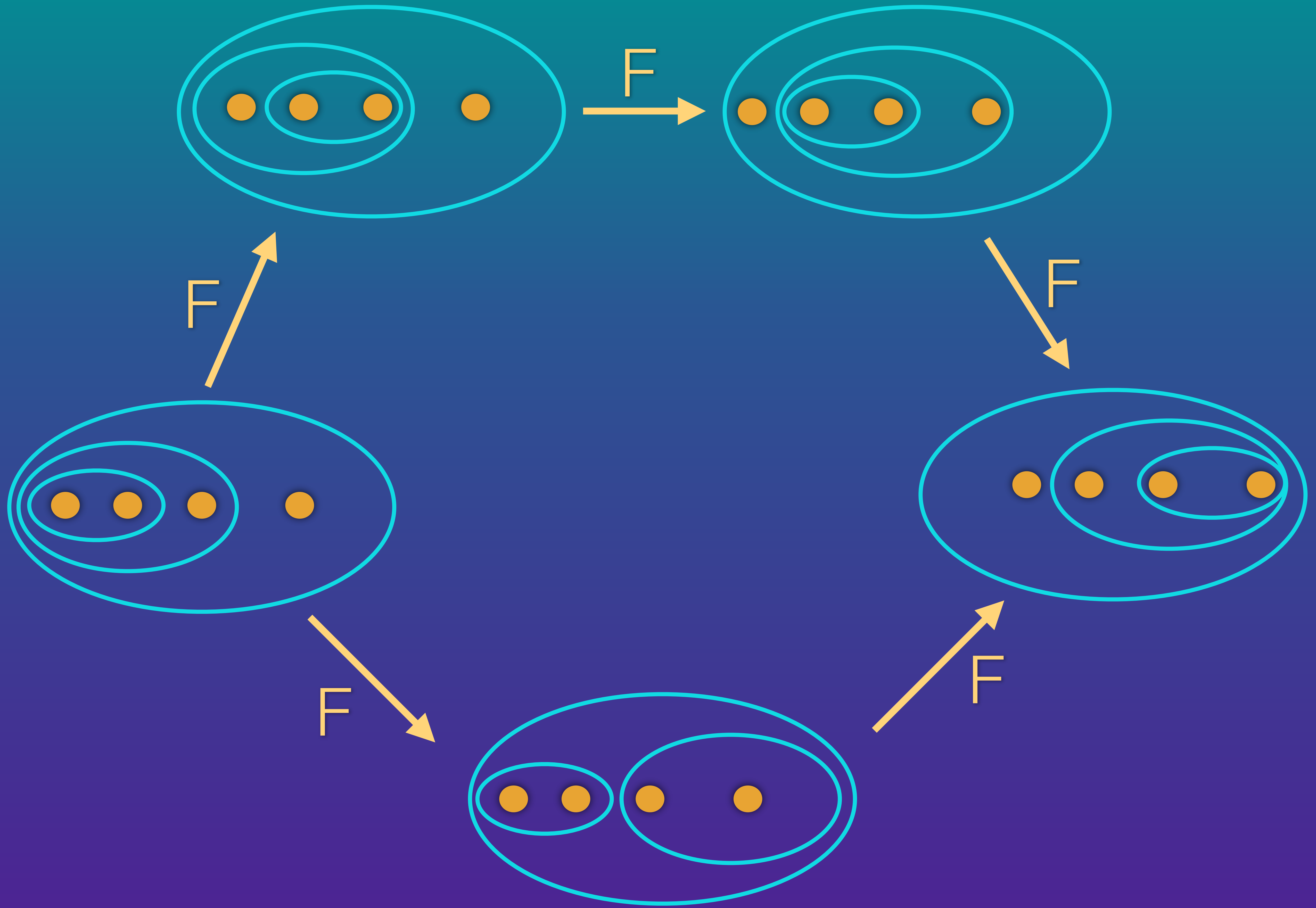


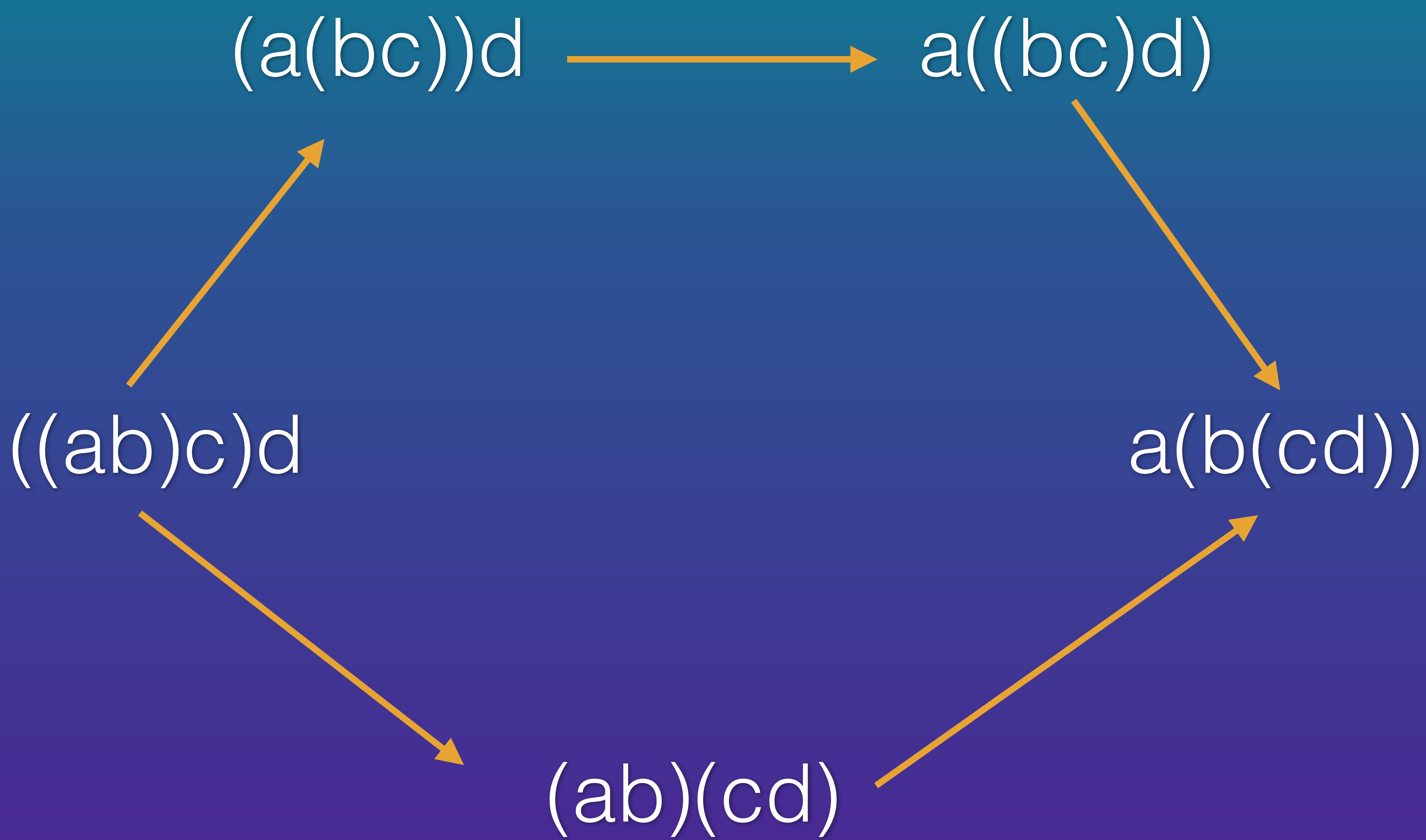




Answer: They should be linear combinations of the previous ones.

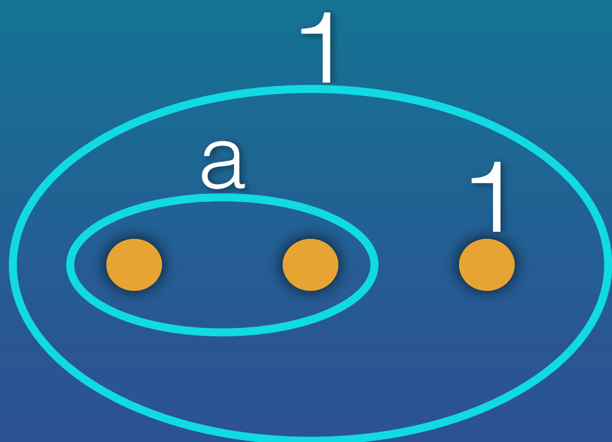
# Consistency Conditions



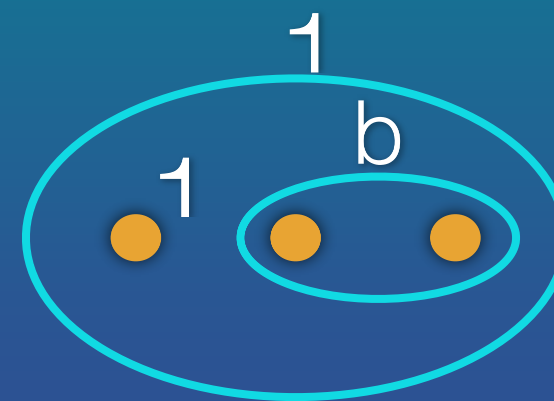

F
F
F
F
F
F


(a(bc))d
a((bc)d)
((ab)c)d
a(b(cd))
(ab)(cd)

# 5-Consistency Relations?




$$= \sum_b F_{ab}$$



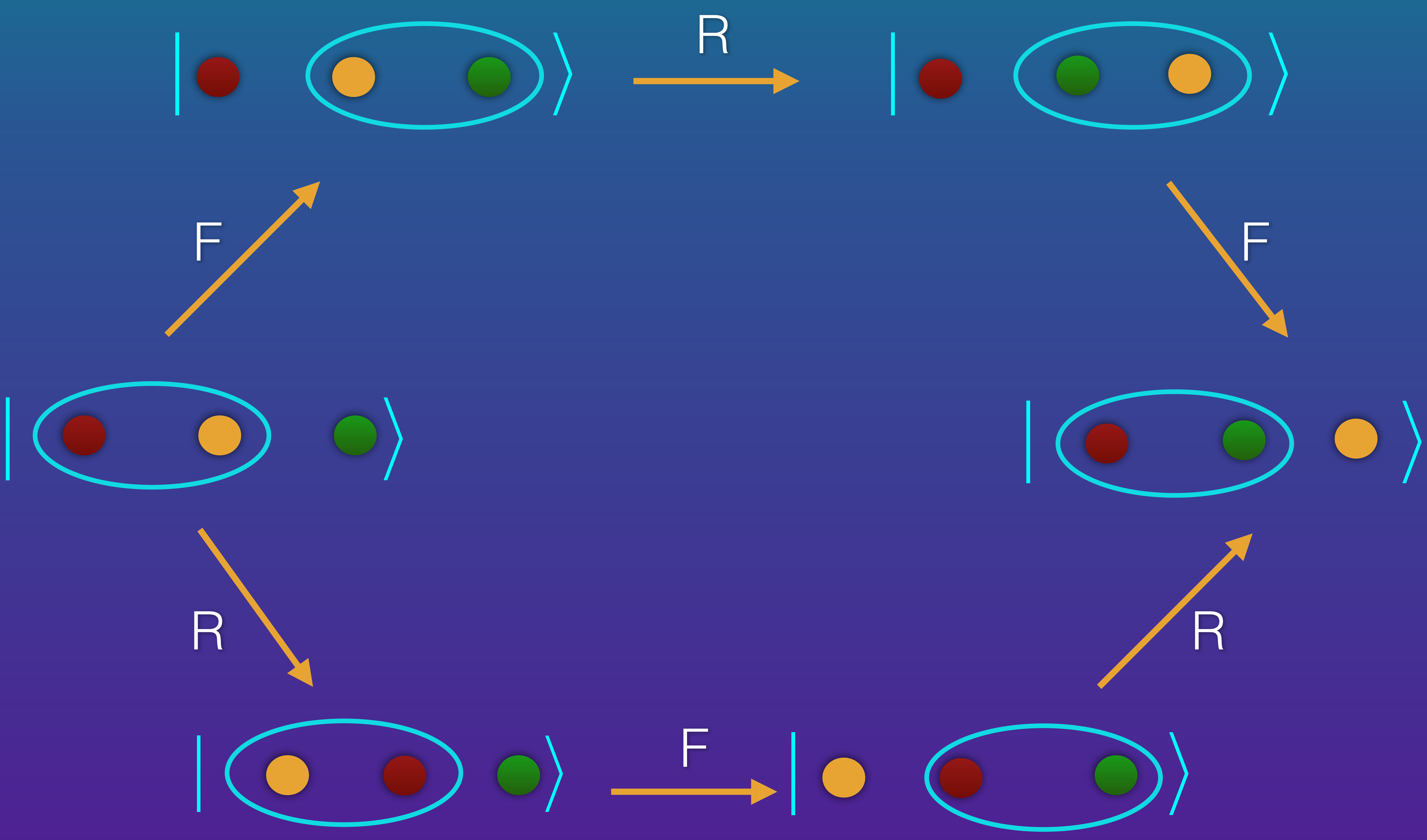

$$F = \begin{pmatrix} \phi & \sqrt{\phi} \\ \sqrt{\phi} & -\phi \end{pmatrix}$$

$$\phi = \frac{\sqrt{5} - 1}{2}$$

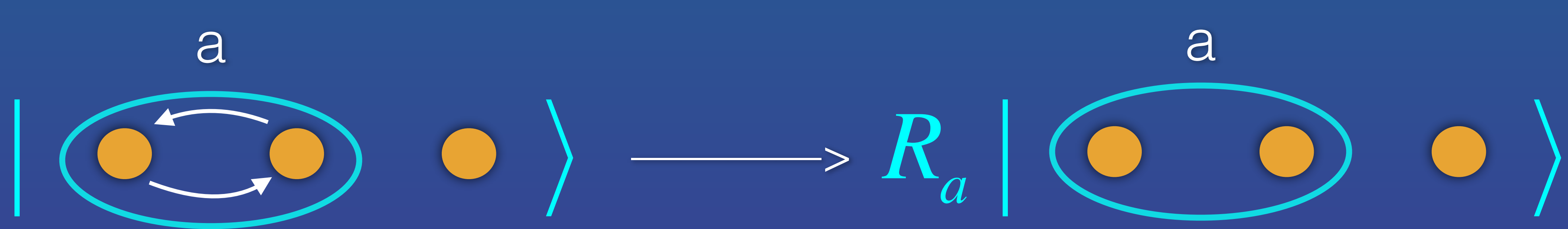
# Braiding


We can do this in two different ways.

The charges obey certain fusion and braiding rules.

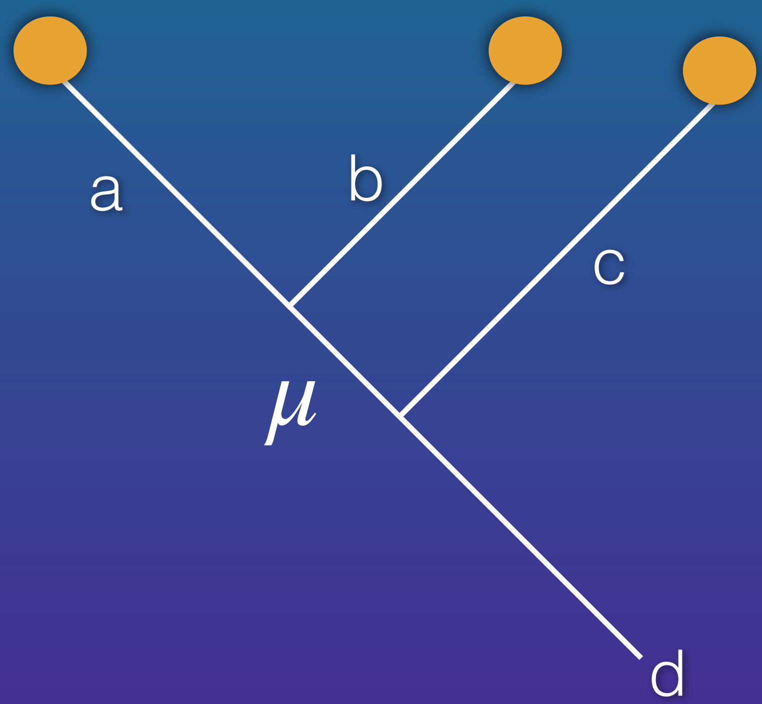


# The braiding matrix

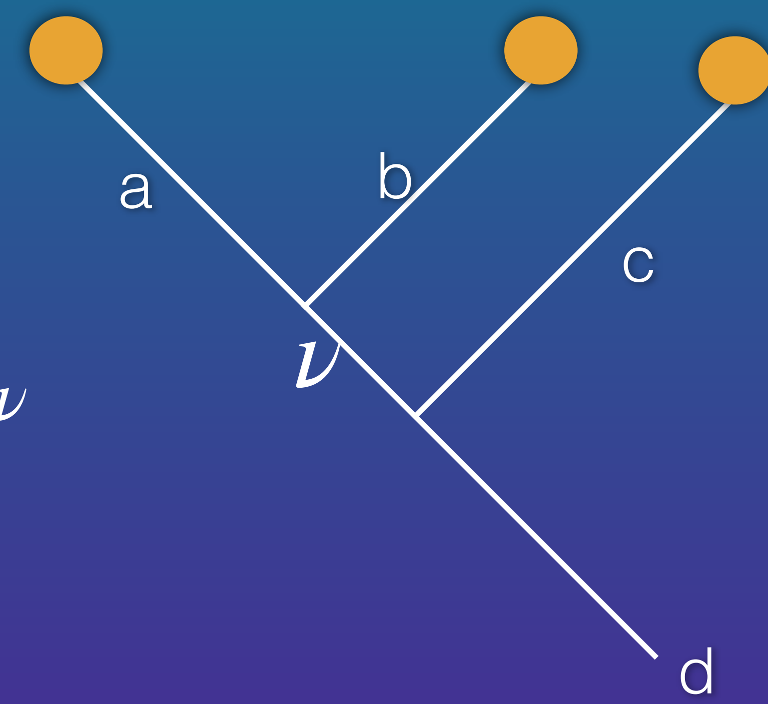



$$R = \begin{pmatrix} R_0 & \\ & R_1 \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{5}} & \\ & -e^{\frac{2\pi i}{5}} \end{pmatrix}$$

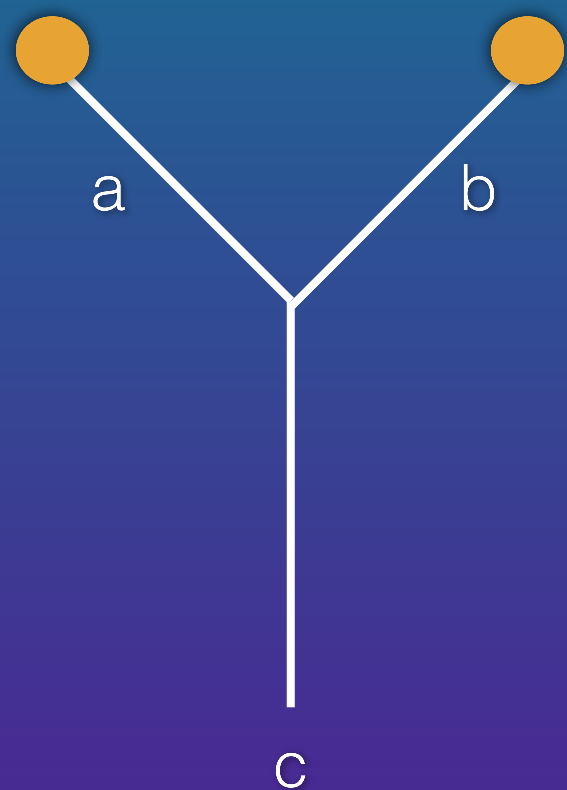
# It is not so simple!



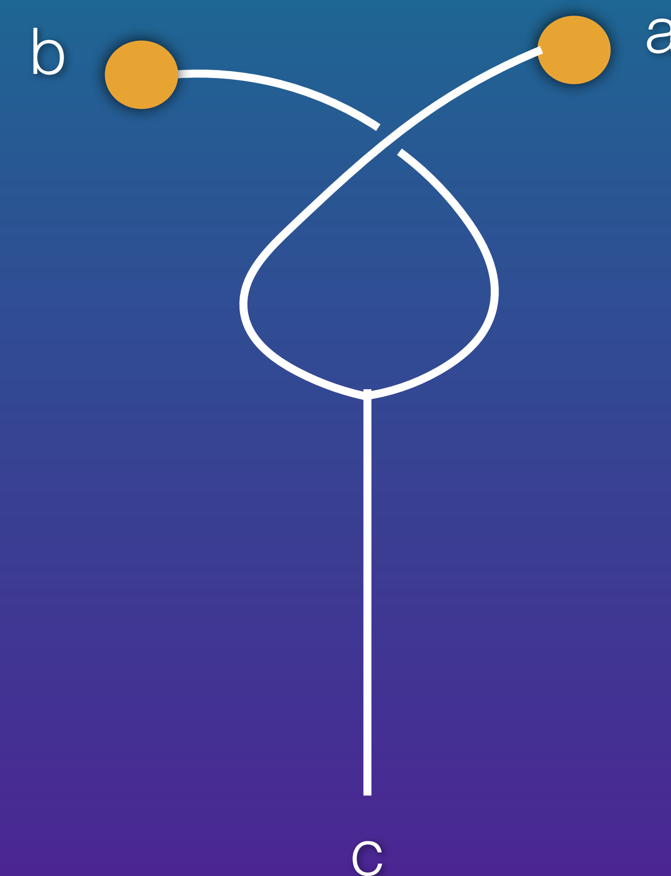

$$= \sum_{\nu} (F^{d}_{abc})_{\mu,\nu}$$



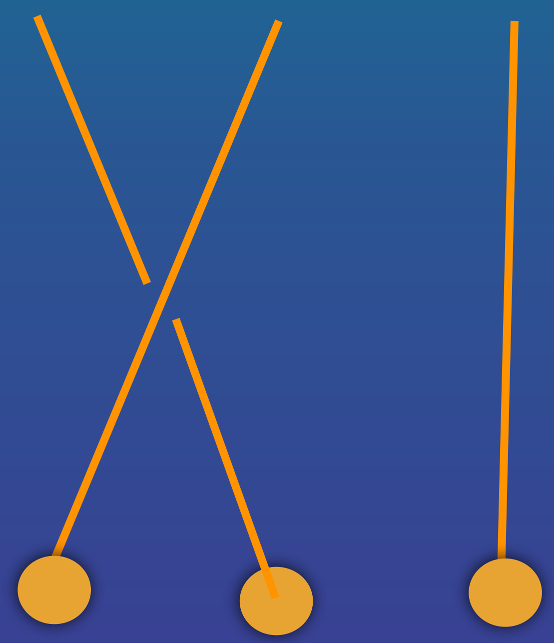
# It is not so simple!

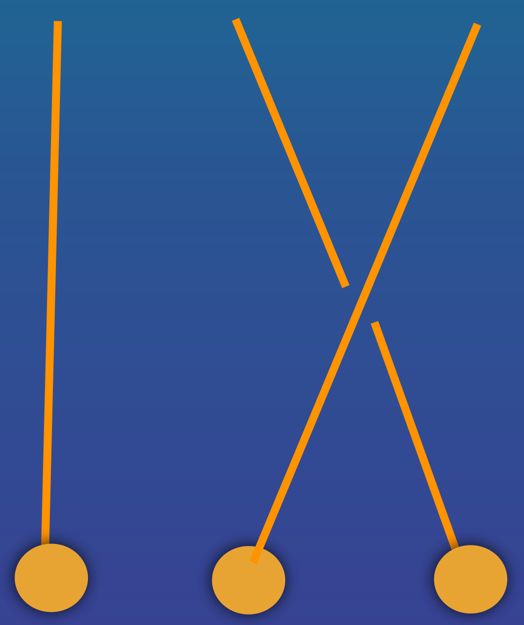



$$= R^{c}_{ab}$$



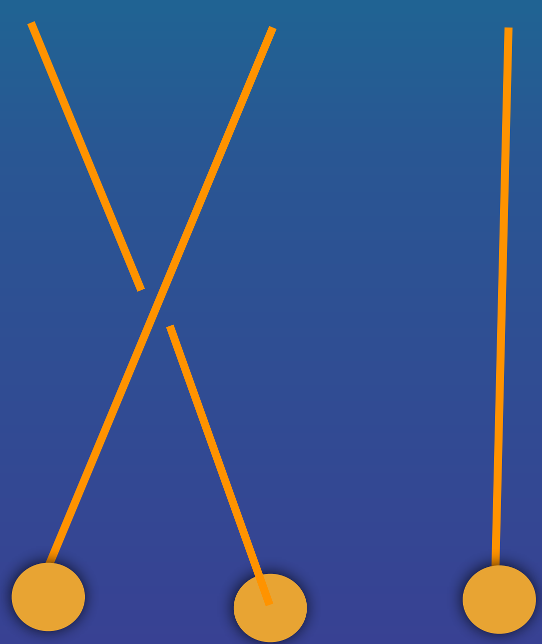
# Quantum Gates



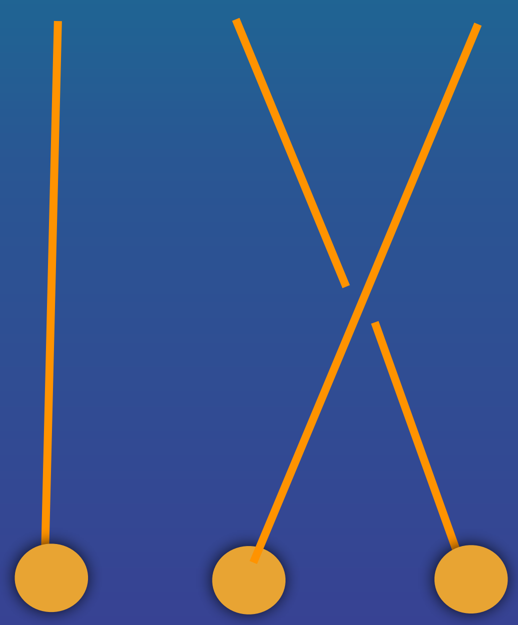


$$\sigma_1 = R$$

$$\sigma_2 = FRF$$



$$\sigma_1 = R = R_z(\frac{3\pi}{10})$$

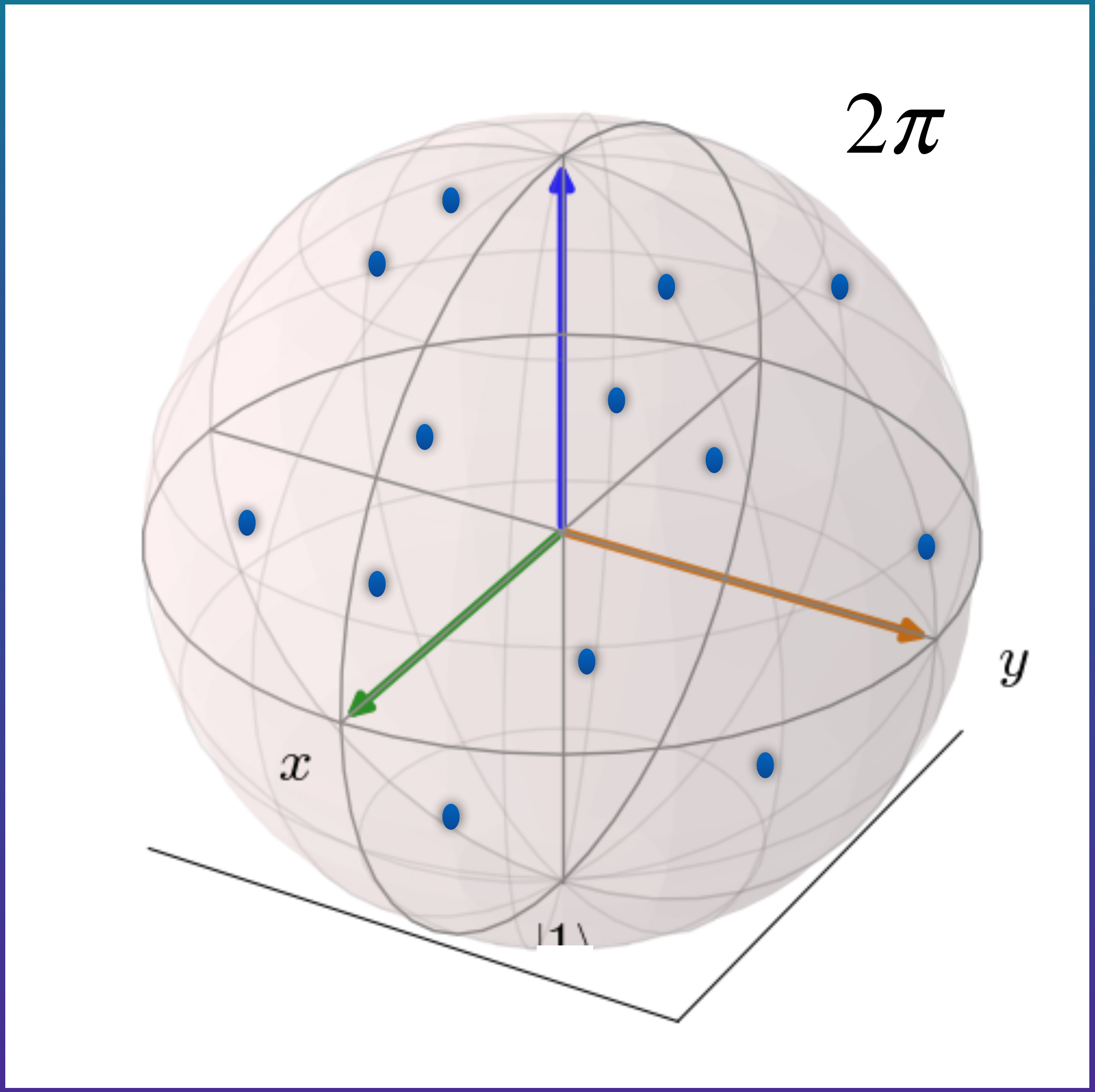


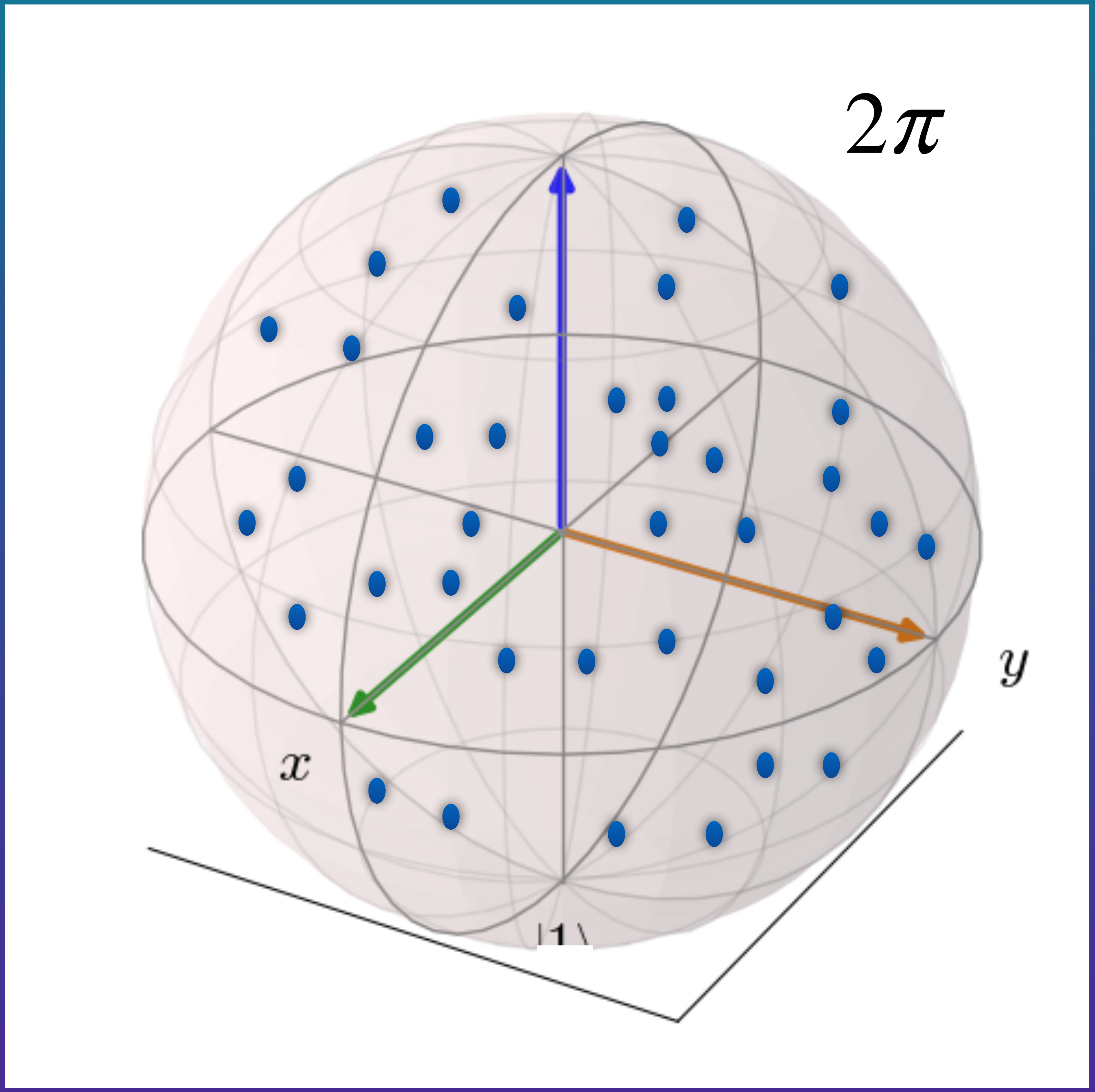
$$\sigma_2 = FRF = R_n(\theta)$$

$$R_z\left(\frac{6\pi}{10}\right)$$

$$R_n(2\theta)$$

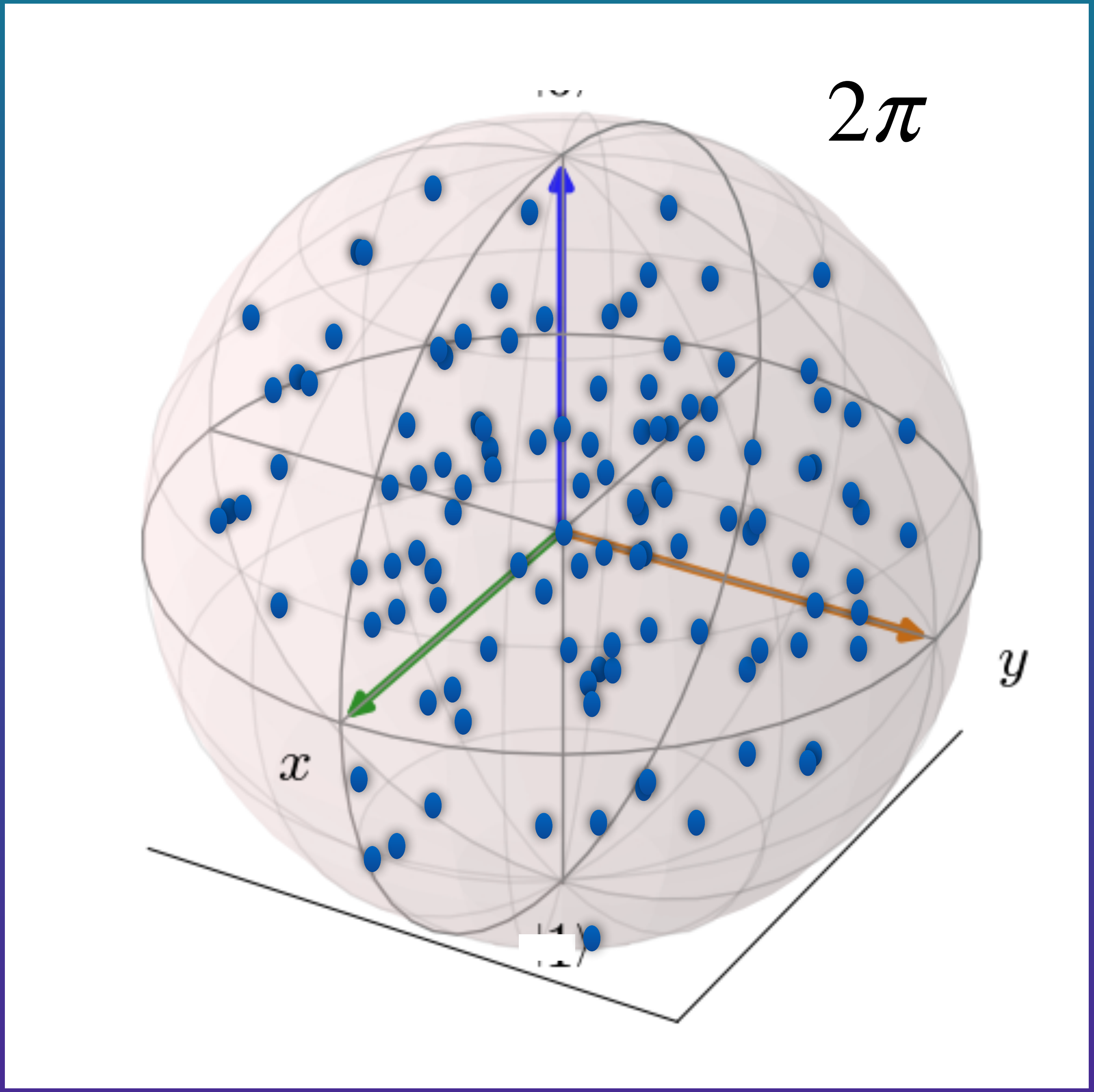
$\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1$



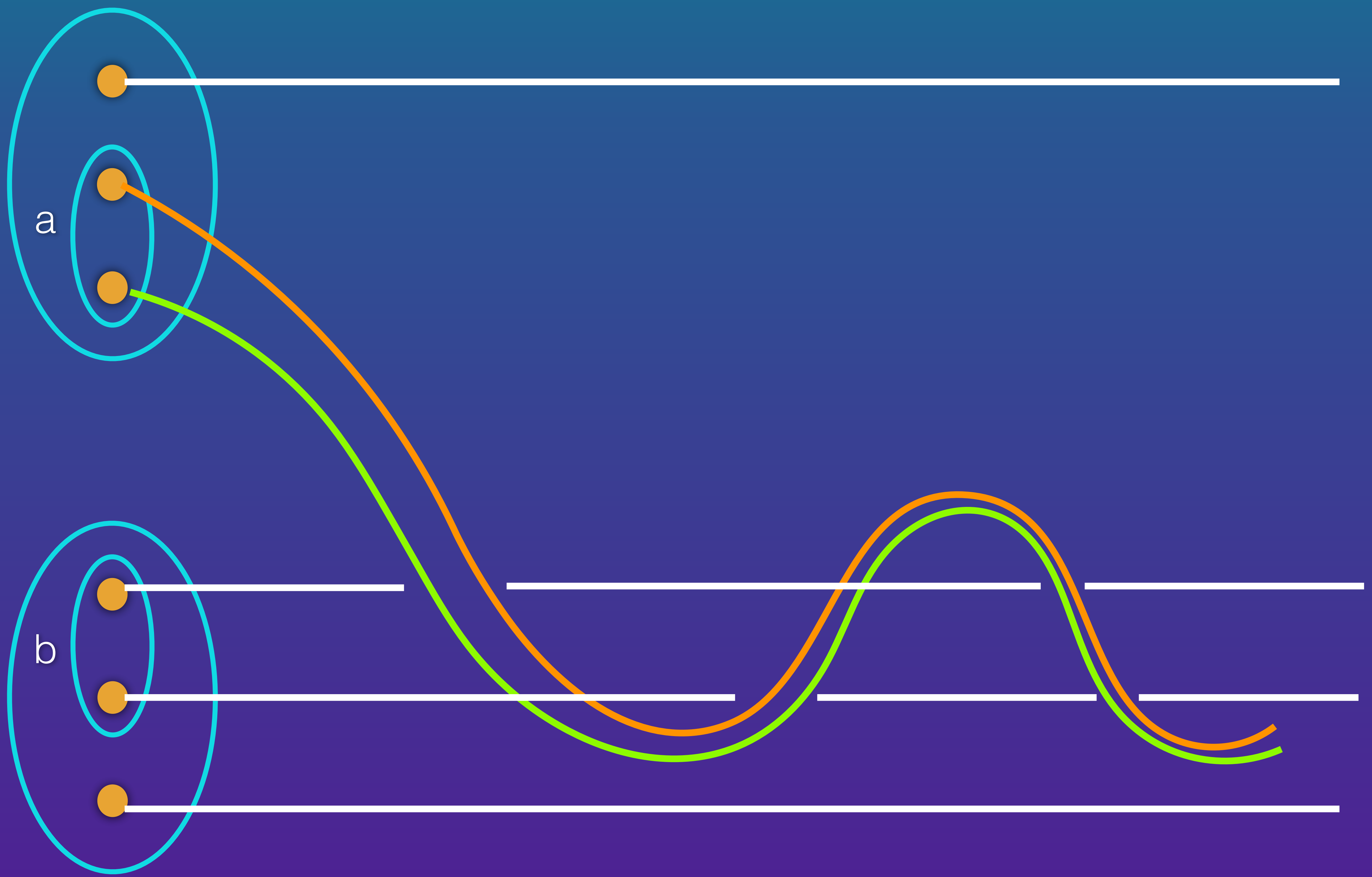



$\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2$

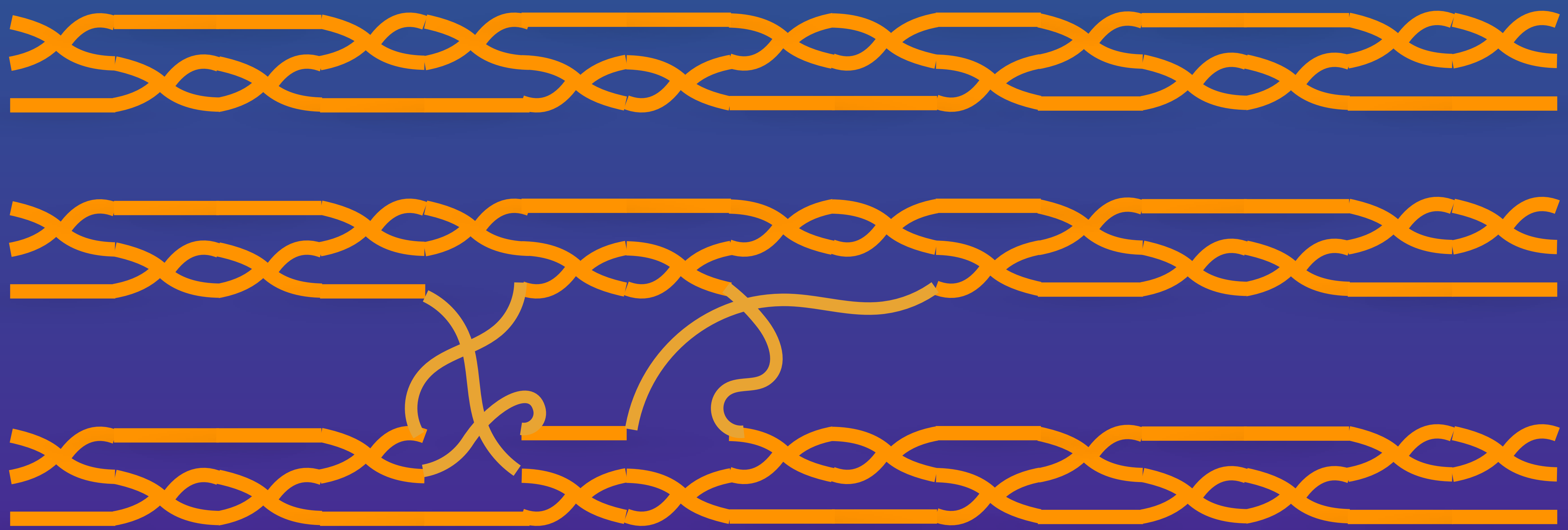
$\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1$



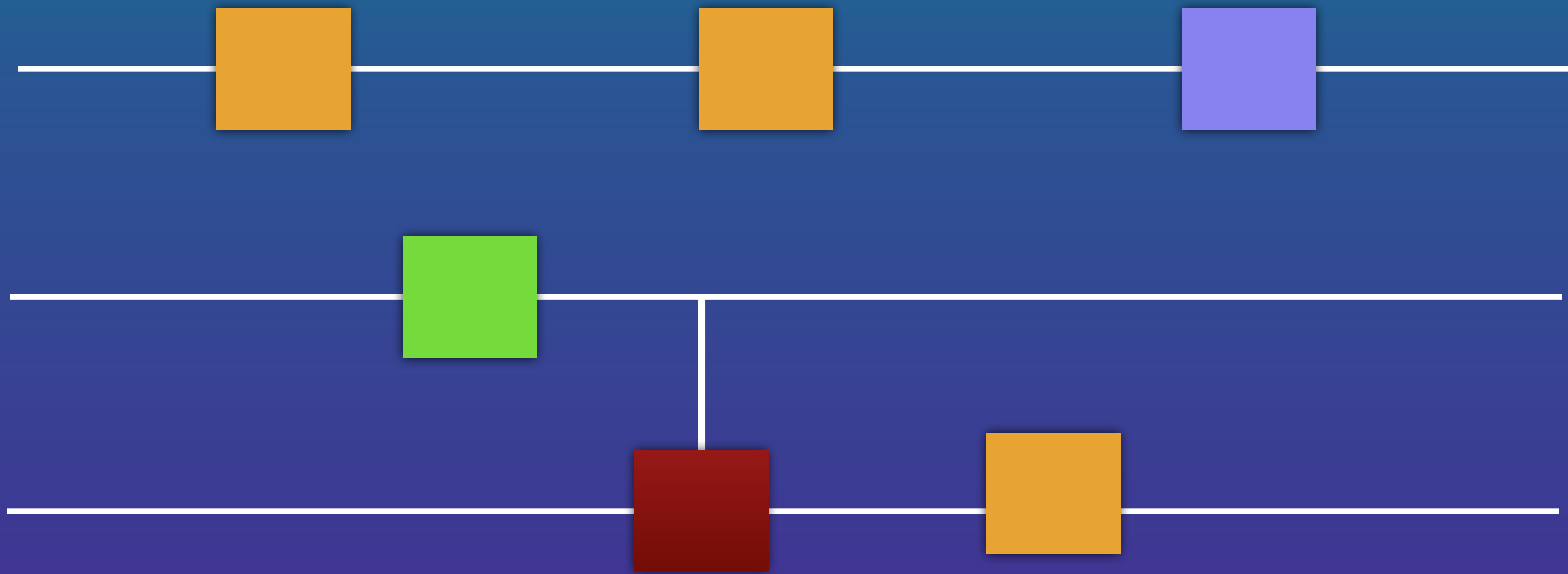
# Two qubit gate
# Control Gate



With one qubit gate and a control two qubit gate, we can do universal quantum computation.



# With one qubit gate and a control two qubit gate, we can do universal quantum computation.



[1] Braid Topologies for Quantum Computation
N. E. Bonesteel, Layla Hormozi, Georgios Zikos, Steven H. Simon, PhysRevLett.95.140503,

# End of part II